SciPost Phys. 19, 014 (2025)

Fermi-liquid corrections to the intrinsic anomalous Hall conductivity of topological metals

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Abstract

We show that topological metals lacking time-reversal symmetry have an intrinsic non-quantised component of the anomalous Hall conductivity which is contributed not only by the Berry phase of quasiparticles on the Fermi surface, but also by Fermi-liquid corrections due to the residual interactions among quasiparticles, the Landau f-parameters. These corrections pair up with those that modify the optical mass with respect to the quasiparticle effective one, or the charge compressibility with respect to the quasiparticle density of states. Our result supports recent claims that the correct expressions for topological observables include vertex corrections besides the topological invariants built just upon the Green's functions. Furthermore, it demonstrates that such corrections are naturally accounted for by Landau's Fermi liquid theory, here extended to the case in which coherence effects between bands crossing the chemical potential and those that are instead away from it may play a crucial role, as in the anomalous Hall conductivity, and have important implications when those metals are on the verge of a doping-driven Mott transition, as we discuss.

Received 2024-08-29
Accepted 2025-06-12
Published 2025-07-16
Check for updates
doi:10.21468/SciPostPhys.19.1.014

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1 Introduction

In 2004, Haldane showed [1], elaborating on earlier works [2–5], that in metals where time-reversal symmetry is broken Landau's Fermi liquid theory [6] must be supplemented by a new, topological ingredient: the quasiparticle Berry phase. Specifically, he demonstrated [1] that the expression for the non-quantised component of the intrinsic anomalous Hall conductivity, e.g., in ferromagnetic metals [7], can be recast as an integral over the quasiparticle Fermi surface, as later confirmed in *ab initio* calculations [8], and in agreement with the spirit of Landau's Fermi liquid theory [6].

However, Haldane's work leaves open an important issue that we now discuss. Landau's energy functional for quasiparticles [6] includes a quasiparticle energy, which we hereafter denote as the Hamiltonian matrix $\hat{H}_*(\mathbf{k})$ having in mind a multiband Fermi liquid, and a residual interaction among quasiparticles, defined in terms of the so-called Landau f-parameters that we write as a tensor $\hat{f}(\mathbf{k},\mathbf{k}')$ in band space. It is tempting to assume that the quasiparticle topology is encoded just in the quasiparticle Hamiltonian $\hat{H}_*(\mathbf{k})$ and its Bloch eigenstates $|\psi_n(\mathbf{k})\rangle$, a relation that was already put forth by Haldane [1]. Nonetheless, there is evidence that such reasonable choice may not be correct [9–16], namely that the Berry curvature calculated through the quasiparticle Bloch states $|\psi_n(\mathbf{k})\rangle$ not necessarily yields the anomalous Hall conductivity. For instance, the authors of Ref. [9] compute the electromagnetic response of a Fermi liquid endowed with a finite Berry curvature through a linearised kinetic equation. They found corrections to the anomalous Hall coefficient, which they identify as coming from an electric dipole moment of the quasiparticles.

It is legitimate to wonder whether this evidence is still compatible with the formal, microscopic derivation of Landau's Fermi liquid theory [17], namely, if the latter does account for Fermi liquid corrections to the anomalous Hall conductivity. This is precisely the aim of the present study.

The structure of the paper is the following. In Sec. 2 we introduce the multi-band formalism of Landau's Fermi-liquid theory and we derive the Hall conductivity tensor, including the Fermi liquid corrections due to the residual interactions between the quasiparticles. In Sec. 3 we specialize our discussion to a model topological metal with broken time-reversal symmetry, where we can explicitly verify that the corrections to the anomalous Hall conductivity stem solely from the quasiparticle Fermi surface. In Sec. 4 we explore the implications of these Fermi liquid corrections in the computation of the topological observables, specifically in strongly correlated topological metals on the verge of a doping-driven Mott transition. The appendices are instead devoted to a more formal derivation of Landau's Fermi liquid theory in a multi-band system.

2 Hall conductivity within Landau's Fermi liquid theory

We assume a periodic system of interacting electrons, thus in absence of impurities and their extrinsic contributions to the anomalous Hall effect [7], at sufficiently low temperature to safely discard any quasiparticle decay rate, and, as mentioned, we consider the general case of many bands crossing the chemical potential that can be described within Landau's Fermi-liquid theory. However, we shall deal with this theory following an early observation by Leggett [18]. Indeed, one may realise [18, 19] that Landau's Fermi-liquid theory basically shows [17] that the low-frequency, long-wavelength and low-temperature response functions of the physical electrons correspond to those of a system of interacting quasiparticles treated by the Hartree-Fock (HF) plus the random phase (RPA) approximations, apart from important caveats that we discuss in Appendix A. In Appendix C we present a detailed derivation of Leggett's result [18] generalised to the case of a multi-band Fermi liquid.

Therefore, let us consider quasiparticles described by an interacting Hamiltonian, hereafter setting $\hbar = 1$,

$$H_{\rm qp} = H_0 + H_{\rm int}\,, \tag{1}$$

where

$$H_0 = \sum_{\mathbf{k}} \sum_{\alpha\beta} c^\dagger_{\alpha\mathbf{k}}\, H_{0\,\alpha\beta}(\mathbf{k})\, c_{\beta\mathbf{k}}\,,$$

is a one-body term, e.g., a tight binding Hamiltonian, with $c_{\alpha\mathbf{k}}$ the annihilation operators of the quasiparticles (here α represents both the band index and the spin σ), and $H_{\rm int}$ the interaction, which we do not even need to specify.

Within HF, one replaces the interacting Hamiltonian (1) with the non-interacting one

$$\hat{H}_*(\mathbf{k}) = \hat{H}_0(\mathbf{k}) + \hat{\Sigma}_{\rm HF}\big[\mathbf{k}, \hat{G}_*\big]\,, \tag{2}$$

which is just the quasiparticle Hamiltonian we mentioned above. In (2), $\hat{H}_0(\mathbf{k})$ is the matrix with elements $H_{0\,\alpha\beta}(\mathbf{k})$, $\hat{\Sigma}_{\rm HF}\big[\mathbf{k}, \hat{G}_*\big]$ includes only the Hartree and Fock diagrams of the self-energy functional, and

$$\hat{G}_*(i\epsilon, \mathbf{k}) = \frac{1}{i\epsilon - \hat{H}_*(\mathbf{k})}\,, \tag{3}$$

is the HF Green's function in Matsubara frequencies. RPA is a symmetry-conserving scheme consistent with HF [19], which amounts to calculate response functions using the Green's function (3) and the irreducible scattering vertex in the particle-hole channel defined as

$$\frac{\delta\hat{\Sigma}_{\rm HF}\big[\mathbf{k}, \hat{G}_*\big]}{\delta\hat{G}_*(i\epsilon, \mathbf{k}')} = \hat{f}(\mathbf{k}, \mathbf{k}')\,, \tag{4}$$

thus providing the definition of the Landau f-parameter tensor. Accordingly, the quasiparticle energies $\epsilon_\ell(\mathbf{k})$ and eigenfunctions $|\,\psi_\ell(\mathbf{k})\rangle$, using roman letters to distinguish them from the greek ones that label the original basis, are obtained by diagonalising

$$\begin{aligned} H_{*\alpha\beta}(\mathbf{k}) &= H_{0\alpha\beta}(\mathbf{k}) + \frac{1}{V}\sum_{\mathbf{k}'\mu\nu} f_{\alpha\nu,\mu\beta}(\mathbf{k},\mathbf{k}')\,\langle\, c^\dagger_{\nu\mathbf{k}'}\, c_{\mu\mathbf{k}'}\,\rangle \\ &= H_{0\alpha\beta}(\mathbf{k}) + \frac{T}{V}\sum_{\mathbf{k}'\mu\nu}\sum_{\epsilon} \mathrm{e}^{i\epsilon 0^+}\, f_{\alpha\nu,\mu\beta}(\mathbf{k},\mathbf{k}')\, G_{*\,\mu\nu}(i\epsilon, \mathbf{k}')\,, \end{aligned} \tag{5}$$

where V is the volume, $f_{\alpha\gamma,\delta\beta}(\mathbf{k}, \mathbf{k}')$ are, through (4), the components of $\hat{f}(\mathbf{k}, \mathbf{k}')$, and the expectation values, to be determined self-consistently, are over the HF ground state. The diagonalization is accomplished by a unitary transformation $\hat{U}(\mathbf{k})$,

$$\hat{U}(\mathbf{k})^\dagger\, \hat{H}_*(\mathbf{k})\, \hat{U}(\mathbf{k}) = \hat{\epsilon}_*(\mathbf{k})\,,$$

where $\hat{\epsilon}_*(\mathbf{k})$ is the diagonal matrix with elements $\epsilon_\ell(\mathbf{k})$.

By means of the Ward-Takahashi identities, Noziéres and Luttinger [17] showed that in the so-called static or q-limit, the transferred frequency vanishing before the transferred momentum, the fully-interacting physical current vertex multiplied by the quasiparticle residue, which we hereafter denote as $\hat{\boldsymbol{J}}^{\,q}(\mathbf{k})$, corresponds to $\boldsymbol{\nabla}_{\mathbf{k}}\hat{H}_*(\mathbf{k})$ for the quasiparticles. In the diagonal basis, which we define as HF basis, that correspondence reads

$$\begin{aligned} \hat{\boldsymbol{J}}^{\,q}(\mathbf{k}) &\equiv \hat{U}(\mathbf{k})^\dagger\, \boldsymbol{\nabla}_{\mathbf{k}}\hat{H}_*(\mathbf{k})\,\hat{U}(\mathbf{k}) = \hat{U}(\mathbf{k})^\dagger\, \boldsymbol{\nabla}_{\mathbf{k}}\Big(\hat{U}(\mathbf{k})\,\hat{\epsilon}_*(\mathbf{k})\,\hat{U}(\mathbf{k})^\dagger\Big)\,\hat{U}(\mathbf{k}) \\ &= \boldsymbol{\nabla}_{\mathbf{k}}\,\hat{\epsilon}_*(\mathbf{k}) - \Big[\,\hat{\epsilon}_*(\mathbf{k})\,,\, \hat{U}(\mathbf{k})^\dagger\, \boldsymbol{\nabla}_{\mathbf{k}}\,\hat{U}(\mathbf{k})\Big] \equiv \boldsymbol{\nabla}_{\mathbf{k}}\,\hat{\epsilon}_*(\mathbf{k}) + i\Big[\,\hat{\epsilon}_*(\mathbf{k}), \hat{\mathcal{A}}^{0}(\mathbf{k})\Big]\,, \end{aligned} \tag{6}$$

where the first term is diagonal and the second off-diagonal in the band indices. Here, the hermitian operator $\hat{\mathcal{A}}^0(\mathbf{k}) = i\,\hat{U}(\mathbf{k})^\dagger\,\nabla_{\mathbf{k}}\,\hat{U}(\mathbf{k})$ is the bare Berry connection of the quasiparticles, whose matrix elements are also equal to

$$\mathcal{A}^0_{\ell m}(\mathbf{k}) = i\,\langle \psi_\ell(\mathbf{k}) \mid \nabla_{\mathbf{k}}\,\psi_m(\mathbf{k})\rangle = -i\,\langle \nabla_{\mathbf{k}}\,\psi_\ell(\mathbf{k}) \mid \psi_m(\mathbf{k})\rangle\,, \tag{7}$$

through which the diagonal part of the bare Berry curvature is $\boldsymbol{\Omega}^0_{\ell\ell}(\mathbf{k}) = -\nabla_{\mathbf{k}} \times \mathcal{A}^0_{\ell\ell}(\mathbf{k})$.

The reason why we have reformulated Landau's Fermi-liquid theory in such more complex language is to make it clear that quasiparticles are interacting, though interaction is treated within HF plus RPA. This point becomes important when one wants to compute topological observables. As already mentioned, the first temptation is to use the eigenfunctions $|\,\psi_\ell(\mathbf{k})\rangle$ and calculate the topological invariants as one would do for non-interacting electrons, see (7). However, this procedure might be incorrect just because the quasiparticles do interact with each other.

A more rigorous approach is to directly calculate the anomalous Hall conductivity as dictated by Landau's Fermi liquid theory [17]. In Appendix A we derive the expression of the matrix elements $\sigma^{\mathrm{H}}_{ab} = -\sigma^{\mathrm{H}}_{ba}$, $a \neq b$ labelling the spatial components, of the Hall conductivity tensor, see (A.24), which read, explicitly,

$$\begin{aligned}
\sigma^{\mathrm{H}}_{ab} &= -i\,\frac{e^2}{V}\sum_{\mathbf{k}}\sum_{\ell\neq m} J^{\omega}_{a\,\ell m}(\mathbf{k})\,\frac{f\big(\epsilon_\ell(\mathbf{k})\big) - f\big(\epsilon_m(\mathbf{k})\big)}{\big(\epsilon_m(\mathbf{k}) - \epsilon_\ell(\mathbf{k})\big)^2}\,J^{\omega}_{b\,m\ell}(\mathbf{k}) \\
&= -i\,\frac{e^2}{V}\sum_{\mathbf{k}}\sum_{\ell\neq m} f\big(\epsilon_\ell(\mathbf{k})\big)\,\frac{J^{\omega}_{a\,\ell m}(\mathbf{k})J^{\omega}_{b\,m\ell}(\mathbf{k}) - J^{\omega}_{b\,\ell m}(\mathbf{k})J^{\omega}_{a\,m\ell}(\mathbf{k})}{\big(\epsilon_m(\mathbf{k}) - \epsilon_\ell(\mathbf{k})\big)^2} \\
&\equiv \frac{e^2}{V}\sum_{\mathbf{k}\ell} f\big(\epsilon_\ell(\mathbf{k})\big)\,\epsilon_{abc}\,\Omega_{c\,\ell\ell}(\mathbf{k})\,,
\end{aligned} \tag{8}$$

where $f(x)$ is the Fermi distribution function, and

$$\boldsymbol{\Omega}_{\ell\ell}(\mathbf{k}) = -i\sum_{m\neq\ell}\frac{\boldsymbol{J}^{\omega}_{\ell m}(\mathbf{k}) \times \boldsymbol{J}^{\omega}_{m\ell}(\mathbf{k})}{\big(\epsilon_m(\mathbf{k}) - \epsilon_\ell(\mathbf{k})\big)^2}\,, \tag{9}$$

is therefore the true quasiparticle Berry curvature. The dynamic, ω-limit of the current vertex, the opposite of the q-limit, is related to (6) through (A.21), specifically,

$$\boldsymbol{J}^{\omega}_{\ell m}(\mathbf{k}) = \boldsymbol{J}^{q}_{\ell m}(\mathbf{k}) - \frac{1}{V}\sum_{\mathbf{k}'n}\Gamma^{\omega}_{\ell n,nm}(\mathbf{k}',\mathbf{k})\,\nabla_{\mathbf{k}'}\,f\big(\epsilon_n(\mathbf{k}')\big)\,, \tag{10}$$

where Γ^{ω} is the ω-limit of the reducible scattering vertex (A.1). The term in (9) with the bare current vertices, i.e., $\Gamma^{\omega} = 0$ in (10), reproduces the bare Berry curvature $\boldsymbol{\Omega}^0_{\ell\ell}(\mathbf{k})$, while the additional terms correspond to the desired Fermi liquid corrections, and indeed derive, see the second term in (10), from the quasiparticle Fermi surface. It is therefore reasonable to argue that also the corrections to the anomalous Hall conductivity come from the quasiparticle Fermi surface, though this conclusion seems not immediately obvious looking at (8) and (10). Indeed, it may seem counterintuitive that a quantity evaluated within the Landau's theory of Fermi liquids could have contributions from bands that do not cross the chemical potential. However, we believe such a paradox to be merely apparent. In the next section, we analyze a specific example and show explicitly that the above surmise is true.

Nonetheless, we emphasize that, while the corrections do derive from quasiparticles at the Fermi surface, equation (8) accounts also for cases where there still are occupied bands that contribute to the anomalous Hall conductivity. It might at first look odd that the contribution of an occupied band may not be quantized because of the interaction with bands crossing the chemical potential, but that does not contradict any physical principle as long as the contribution becomes again quantized when no band cross anymore the chemical potential, which is clearly the case here, see (10).

3 A toy model calculation

We consider the Bernevig, Hughes and Zhang (BHZ) model [20] for a quantum spin-Hall insulator on a square lattice. In particular, we assume the model with full spin polarisation, only spin up bands being occupied, and at density $n = 1+\delta$, in which case the model describes a topological metal with broken time-reversal symmetry.

The Hamiltonian for the spin-up quasiparticles is assumed to be

$$\begin{aligned}\hat{H}_*(\mathbf{k}) &= \big(\epsilon(\mathbf{k})-\mu\big)\,\sigma_0 + \big(M - t(\mathbf{k})\big)\,\sigma_3 + \lambda(\mathbf{k})\,\sin k_x\,\sigma_1 - \lambda(\mathbf{k})\,\sin k_y\,\sigma_2 \\ &= \big(\epsilon(\mathbf{k})-\mu\big)\,\sigma_0 + \boldsymbol{B}(\mathbf{k})\cdot\boldsymbol{\sigma}\,,\end{aligned} \tag{11}$$

where μ crosses either the valence or the conduction bands, the identity, σ_0, and the Pauli matrices σ_a, $a = 1,2,3$, act on the orbital space, and

$$\begin{aligned}\boldsymbol{B}(\mathbf{k}) &= \big(\lambda(\mathbf{k})\,\sin k_x, -\lambda(\mathbf{k})\,\sin k_y, M - t(\mathbf{k})\big) \\ &= B(\mathbf{k})\Big(\cos\phi(\mathbf{k})\,\sin\theta(\mathbf{k}), \sin\phi(\mathbf{k})\,\sin\theta(\mathbf{k}), \cos\theta(\mathbf{k})\Big) \\ &= B(\mathbf{k})\Big(\sin\theta(\mathbf{k})\,\boldsymbol{v}_2(\mathbf{k}) + \cos\theta(\mathbf{k})\,\boldsymbol{v}_3(\mathbf{k})\Big)\,.\end{aligned} \tag{12}$$

Here, $\boldsymbol{v}_2(\mathbf{k}) = (\cos\phi(\mathbf{k}), \sin\phi(\mathbf{k}), 0)$ and $\boldsymbol{v}_3(\mathbf{k}) = (0,0,1)$ are orthogonal unit vectors that form a basis together with $\boldsymbol{v}_1(\mathbf{k}) = \boldsymbol{v}_2(\mathbf{k}) \times \boldsymbol{v}_3(\mathbf{k})$. This, in turn, implies that σ_0 and $\boldsymbol{v}_a(\mathbf{k})\cdot\boldsymbol{\sigma}$, $a = 1,2,3$, form a basis of 2×2 matrices. To simplify the notations, in what follows we use the definition $\boldsymbol{v}_0(\mathbf{k})\cdot\boldsymbol{\sigma} :== \sigma_0$.

The quasiparticle Hamiltonian (11) is assumed to be invariant under inversion $\mathcal{I}$, $\mathbf{k}\to-\mathbf{k}$ and $\sigma_a \to -\sigma_a$ for $a = 1,2$, and fourfold rotations C_4, $k_x \to k_y \wedge k_y \to -k_x$ and $\sigma_1 \to -\sigma_2 \wedge \sigma_2 \to \sigma_1$. This requires that the parameters $\epsilon(\mathbf{k})$, $t(\mathbf{k})$ and $\lambda(\mathbf{k})$ in (11) are invariant under both $\mathcal{I}$ and C_4. In addition, in order to clearly distinguish μ and M from, respectively, $\epsilon(\mathbf{k})$ and $t(\mathbf{k})$, we assume that the latter average out at zero over the Brillouin zone.

The Hamiltonian $\hat{H}_*(\mathbf{k})$ is diagonalised by the unitary transformation

$$\hat{U}(\mathbf{k}) = \mathrm{e}^{i\,\frac{\theta(\mathbf{k})}{2}\,\boldsymbol{v}_1(\mathbf{k})\cdot\boldsymbol{\sigma}} = \cos\frac{\theta(\mathbf{k})}{2} + i\,\sin\frac{\theta(\mathbf{k})}{2}\,\boldsymbol{v}_1(\mathbf{k})\cdot\boldsymbol{\sigma}\,,$$

namely $\hat{U}(\mathbf{k})^\dagger\,\hat{H}_*(\mathbf{k})\,\hat{U}(\mathbf{k}) = (\epsilon(\mathbf{k})-\mu)\,\sigma_0 + B(\mathbf{k})\,\sigma_3$. We readily find that

$$\begin{aligned}\hat{\mathcal{A}}^0(\mathbf{k}) &= i\,\hat{U}(\mathbf{k})^\dagger\,\boldsymbol{\nabla}_\mathbf{k}\,\hat{U}(\mathbf{k}) = -\frac{\boldsymbol{\nabla}_\mathbf{k}\,\theta(\mathbf{k})}{2}\,\boldsymbol{v}_1(\mathbf{k})\cdot\boldsymbol{\sigma} \\ &\quad - \frac{\boldsymbol{\nabla}_\mathbf{k}\,\phi(\mathbf{k})}{2}\,\sin\theta(\mathbf{k})\,\boldsymbol{v}_2(\mathbf{k})\cdot\boldsymbol{\sigma} - \boldsymbol{\nabla}_\mathbf{k}\,\phi(\mathbf{k})\,\sin^2\frac{\theta(\mathbf{k})}{2}\,\boldsymbol{v}_3(\mathbf{k})\cdot\boldsymbol{\sigma} \\ &= \sum_{a=1}^{3}\mathcal{A}^0_a(\mathbf{k})\,\boldsymbol{v}_a(\mathbf{k})\cdot\boldsymbol{\sigma}\,.\end{aligned} \tag{13}$$

We can make a similar expansion for the currents,

$$\hat{J}^q(\mathbf{k}) = \sum_{a=0}^{3} J^q_a(\mathbf{k})\,\boldsymbol{v}_a(\mathbf{k})\cdot\boldsymbol{\sigma}\,, \qquad \hat{J}^\omega(\mathbf{k}) = \sum_{a=0}^{3} J^\omega_a(\mathbf{k})\,\boldsymbol{v}_a(\mathbf{k})\cdot\boldsymbol{\sigma}\,,$$

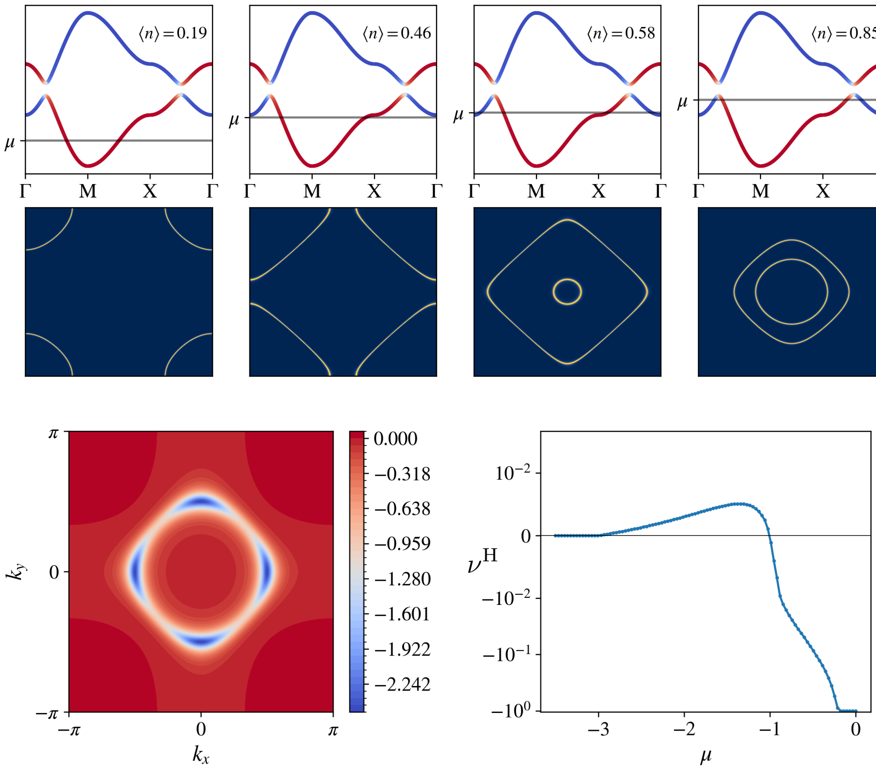

Figure 1: Top panel: Band structures of (11) with $\epsilon(\mathbf{k}) = 0$, $M = 1$, $t(\mathbf{k}) = \cos k_x + \cos k_y$ and $\lambda(\mathbf{k}) = 0.2$ (top), and Fermi surfaces (bottom) for different values of μ corresponding to hole doping. Bottom panel: distribution in momentum space of the bare Berry curvature $\Omega^0(\mathbf{k})$ of the valence band (left); integral ν^{H} of the bare Berry curvature in units of 2π over the occupied Fermi volume (right) as function of $\mu < 0$. When $2|\mu|$ is smaller than the gap, $\nu^{\mathrm{H}} = -1$ keeps its quantised value, otherwise it deviates reaching zero when the valence band empties, $\mu = -3$. In reality, ν^{H} crosses zero also when the Fermi surface changes character, from hole-like to electron-like. We also note that $\Omega^0(\mathbf{k})$ of the valence band is peaked in magnitude at the top of the band, when the orbital character changes. This explains the fast decrease in magnitude upon hole doping.

where $\boldsymbol{J}^q_0(\mathbf{k}) = \boldsymbol{\nabla}_{\mathbf{k}}\, \epsilon(\mathbf{k})$, $\boldsymbol{J}^q_3(\mathbf{k}) = \boldsymbol{\nabla}_{\mathbf{k}}\, B(\mathbf{k})$ and $\boldsymbol{J}^q_a(\mathbf{k}) = 2\, B(\mathbf{k})\, \epsilon_{ab3}\, \mathcal{A}^0_b(\mathbf{k})$ for $a = 1, 2$. In the top panel of Fig. 1 we show the band structure of (11) at $\epsilon(\mathbf{k}) = 0$, $M = 1$, $t(\mathbf{k}) = \cos k_x + \cos k_y$ and $\lambda(\mathbf{k}) = 0.2$, indicating the chemical potential μ for the different hole doping levels and the corresponding Fermi surfaces. In the bottom panel, we draw the momentum distribution of the bare Berry curvature of the valence band, as well as its integral in units of 2π over the occupied Fermi volume as function of the chemical potential $\mu < 0$.

We note that $\boldsymbol{v}_a(\mathbf{k}) \cdot \boldsymbol{\sigma}$, $a = 0, \ldots, 3$, are invariant under both $\mathcal{I}$ and C_4, and $\boldsymbol{\nabla}_{\mathbf{k}}\, \phi(\mathbf{k})$ as well as $\boldsymbol{\nabla}_{\mathbf{k}}\, \theta(\mathbf{k})$ transform as conventional vectors. Since the f-parameters contribute to the quasiparticle Hamiltonian (11) through the mean-field decoupling, the expression of $\boldsymbol{B}(\mathbf{k})$ in (12) implies that the most general $\hat{f}$-tensor in the original basis must be of the form

$$\hat{f}(\mathbf{k}, \mathbf{k}') = \sum_{a,b=0,2,3} f_{ab}(\mathbf{k}, \mathbf{k}') \big(\boldsymbol{v}_a(\mathbf{k}) \cdot \boldsymbol{\sigma}\big) \otimes \big(\boldsymbol{v}_b(\mathbf{k}') \cdot \boldsymbol{\sigma}\big), \tag{14}$$

thus lacking terms that involve $\boldsymbol{v}_1(\mathbf{k}) \cdot \boldsymbol{\sigma}$, and where symmetry only requires that the expansion coefficients are invariant under $\mathcal{I}$ and C_4. Upon rotation in the HF basis, (14) maintains the same form with modified coefficients, which we still denote as $f_{ab}(\mathbf{k},\mathbf{k}')$ for sake of simplicity. We can further identify two distinct tensors, $\hat{f}^{\,0}(\mathbf{k},\mathbf{k}') = \hat{f}^{\,0}(-\mathbf{k},\mathbf{k}') = \hat{f}^{\,0}(\mathbf{k},-\mathbf{k}')$ and $\hat{f}^{\,1}(\mathbf{k},\mathbf{k}') = -\hat{f}^{\,1}(-\mathbf{k},\mathbf{k}') = -\hat{f}^{\,1}(\mathbf{k},-\mathbf{k}')$, using the same notation as in conventional single-band Fermi liquids [21]. It is precisely $\hat{f}^{\,1}(\mathbf{k},\mathbf{k}')$ that may contribute to (10), which, without loss of generality, can be taken in the HF basis of the form

$$\hat{f}^{\,1}(\mathbf{k},\mathbf{k}') = \sum_{a,b=0,2,3} f^{1}_{ab}(\mathbf{k},\mathbf{k}')\, \boldsymbol{J}^{q}_{a}(\mathbf{k})\cdot \boldsymbol{J}^{q}_{b}(\mathbf{k}')\big(\boldsymbol{v}_a(\mathbf{k})\cdot\boldsymbol{\sigma}\big)\otimes\big(\boldsymbol{v}_b(\mathbf{k}')\cdot\boldsymbol{\sigma}\big),$$

where, for consistency, $f^{1}_{ab}(\mathbf{k},\mathbf{k}') = f^{1}_{ab}(-\mathbf{k},\mathbf{k}') = f^{1}_{ab}(\mathbf{k},-\mathbf{k}')$ vary in momentum space orthogonally to the corresponding currents, i.e.,

$$\boldsymbol{J}^{q}_{a}(\mathbf{k})\cdot\boldsymbol{\nabla}_{\mathbf{k}} f^{1}_{ab}(\mathbf{k},\mathbf{k}') = \boldsymbol{\nabla}_{\mathbf{k}'} f^{1}_{ab}(\mathbf{k},\mathbf{k}')\cdot \boldsymbol{J}^{q}_{b}(\mathbf{k}') = 0\,, \tag{15}$$

for $a,b=0,2,3$. In reality, only the components $f^{1}_{ab}(\mathbf{k},\mathbf{k}')$, $a=0,2,3$ and $b=0,3$, contribute to (10), in which case (A.1) implies that $\Gamma^{1\omega}_{ab}(\mathbf{k},\mathbf{k}') \equiv f^{1}_{ab}(\mathbf{k},\mathbf{k}')$. It is worth remarking that $\hat{f}(\mathbf{k},\mathbf{k}')$ in (14) contributes to $\hat{H}_*(\mathbf{k})$ in (11) through the HF self-energy, see (5). Therefore, if we reasonably assume that the HF ground state does not carry any current, then the mean-field terms generated by $\hat{f}^{\,1}(\mathbf{k},\mathbf{k}')$ must vanish at self-consistency, which requires, at the very least, non-singular coefficients $f^{1}_{ab}(\mathbf{k},\mathbf{k}')$. This observation will be useful later on.

We further define the diagonal matrix $\hat{\mathcal{P}}(\mathbf{k})$ with elements $f\big(\epsilon_\ell(\mathbf{k})\big)$, which we can write as

$$\begin{aligned}\hat{\mathcal{P}}(\mathbf{k}) &= \frac{f\big(\epsilon_1(\mathbf{k})\big)+f\big(\epsilon_2(\mathbf{k})\big)}{2}\,\boldsymbol{v}_0(\mathbf{k})\cdot\boldsymbol{\sigma} + \frac{f\big(\epsilon_1(\mathbf{k})\big)-f\big(\epsilon_2(\mathbf{k})\big)}{2}\,\boldsymbol{v}_3(\mathbf{k})\cdot\boldsymbol{\sigma}\\ &\equiv \mathcal{P}_0(\mathbf{k})\,\boldsymbol{v}_0(\mathbf{k})\cdot\boldsymbol{\sigma} + \mathcal{P}_3(\mathbf{k})\,\boldsymbol{v}_3(\mathbf{k})\cdot\boldsymbol{\sigma}\,,\end{aligned}$$

where the conduction band 1 corresponds to $\sigma_3=+1$ with energy $\epsilon_1(\mathbf{k}) = \epsilon(\mathbf{k})-\mu+B(\mathbf{k})$, and the valence band 2 to $\sigma_3=-1$ and $\epsilon_2(\mathbf{k}) = \epsilon(\mathbf{k})-\mu-B(\mathbf{k})$. With those definitions, (10) transforms into an equation for each component in the matrix basis, specifically, and exploiting the spatial symmetries,

$$\begin{aligned}\boldsymbol{J}^{\omega}_{a}(\mathbf{k}) &= \boldsymbol{J}^{q}_{a}(\mathbf{k}) - \big(1-\delta_{a,1}\big)\frac{2}{V}\sum_{\mathbf{k}'}\sum_{b=0,3} f^{1}_{ab}(\mathbf{k},\mathbf{k}')\,\boldsymbol{\nabla}_{\mathbf{k}'}\mathcal{P}_b(\mathbf{k}')\Big(\boldsymbol{J}^{q}_{a}(\mathbf{k})\cdot\boldsymbol{J}^{q}_{b}(\mathbf{k}')\Big)\\ &= \boldsymbol{J}^{q}_{a}(\mathbf{k})\left\{1-\big(1-\delta_{a,1}\big)\frac{1}{V}\sum_{\mathbf{k}'}\sum_{b=0,3} f^{1}_{ab}(\mathbf{k},\mathbf{k}')\,\boldsymbol{\nabla}_{\mathbf{k}'}\mathcal{P}_b(\mathbf{k}')\cdot\boldsymbol{J}^{q}_{b}(\mathbf{k}')\right\}\\ &\equiv \boldsymbol{J}^{q}_{a}(\mathbf{k})\left(1+\big(1-\delta_{a,1}\big)\frac{F^{1}_{a}(\mathbf{k})}{2}\right).\end{aligned} \tag{16}$$

The anomalous Hall conductivity $\sigma^{\mathrm{H}} = \sigma^{\mathrm{H}}_{xy}$ in (8) can be simply written as

$$\begin{aligned}\sigma^{\mathrm{H}} &= -i\,\frac{e^2}{V}\sum_{\mathbf{k}}\sum_{a,b=1}^{2}\frac{J^{\omega}_{xa}(\mathbf{k})J^{\omega}_{yb}(\mathbf{k})}{4B(\mathbf{k})^2}\,\mathrm{Tr}\Big(\Big[\mathcal{P}_3(\mathbf{k})\,\boldsymbol{v}_3(\mathbf{k})\cdot\boldsymbol{\sigma},\boldsymbol{v}_a(\mathbf{k})\cdot\boldsymbol{\sigma}\Big]\,\boldsymbol{v}_b(\mathbf{k})\cdot\boldsymbol{\sigma}\Big)\\ &= \frac{e^2}{2V}\sum_{\mathbf{k}}\sum_{a,b=1}^{2}\frac{\mathcal{P}_3(\mathbf{k})}{B(\mathbf{k})^2}\,\epsilon_{ab3}\,\boldsymbol{J}^{\omega}_{a}(\mathbf{k})\times\boldsymbol{J}^{\omega}_{b}(\mathbf{k})\cdot\boldsymbol{v}_3(\mathbf{k})\\ &= -\frac{e^2}{V}\sum_{\mathbf{k}}\mathcal{P}_3(\mathbf{k})\left(1+\frac{F^{1}_{2}(\mathbf{k})}{2}\right)\boldsymbol{\nabla}_{\mathbf{k}}\times\Big(\cos\theta(\mathbf{k})\,\boldsymbol{\nabla}_{\mathbf{k}}\,\phi(\mathbf{k})\Big)\cdot\boldsymbol{v}_3(\mathbf{k})\\ &= \frac{e^2}{V}\sum_{\mathbf{k}}\cos\theta(\mathbf{k})\left(1+\frac{F^{1}_{2}(\mathbf{k})}{2}\right)\boldsymbol{\nabla}_{\mathbf{k}}\mathcal{P}_3(\mathbf{k})\times\boldsymbol{\nabla}_{\mathbf{k}}\,\phi(\mathbf{k})\cdot\boldsymbol{v}_3(\mathbf{k})\\ &\quad + \frac{e^2}{2V}\sum_{\mathbf{k}}\cos\theta(\mathbf{k})\,\mathcal{P}_3(\mathbf{k})\,\boldsymbol{\nabla}_{\mathbf{k}}F^{1}_{2}(\mathbf{k})\times\boldsymbol{\nabla}_{\mathbf{k}}\,\phi(\mathbf{k})\cdot\boldsymbol{v}_3(\mathbf{k})\,.\end{aligned} \tag{17}$$

We note that the first term in the last equality of (17) is a genuine Fermi surface contribution. Concerning the last term, we recall that $\boldsymbol{J}^q_2(\mathbf{k}) \propto \mathcal{A}^0_1(\mathbf{k}) \propto \boldsymbol{\nabla}_\mathbf{k}\,\theta(\mathbf{k})$ and, because of (15), the vector product $\boldsymbol{\nabla}_\mathbf{k}\, F^1_2(\mathbf{k}) \times \boldsymbol{\nabla}_\mathbf{k}\,\phi(\mathbf{k})$ is proportional to $\sin k_x\, \sin k_y$, odd under C_4, which therefore averages out at zero upon summing over $\mathbf{k}$ since $\mathcal{P}_3(\mathbf{k})$ and $\cos\theta(\mathbf{k})$ are both invariant. In conclusion

$$\sigma^{\rm H} = \frac{e^2}{V}\sum_{\mathbf{k}}\left(1+\frac{F^1_2(\mathbf{k})}{2}\right)\cos\theta(\mathbf{k})\;\boldsymbol{\nabla}_\mathbf{k}\,\mathcal{P}_3(\mathbf{k})\times\boldsymbol{\nabla}_\mathbf{k}\,\phi(\mathbf{k})\cdot\boldsymbol{v}_3(\mathbf{k}) \equiv \sigma^{\rm H}_0\left(1+\frac{F^1_2}{2}\right), \tag{18}$$

where $\sigma^{\rm H}_0$ is the bare value, i.e., neglecting the vertex corrections, and F^1_2 is a weighted average over the Fermi surfaces. Therefore, in the model (11) we can explicitly verify that the corrections to the anomalous Hall conductivity only derive from the quasiparticle Fermi surface, as we earlier conjectured. We believe that this result has a more general validity.

For completeness, the Drude weight $D_{xx} = D_{yy} = D$ can be readily found through (A.20),

$$D = D_0\left(1+\frac{F^1}{2}\right), \tag{19}$$

where

$$D_0 = -\frac{e^2}{2V}\sum_{\mathbf{k}}\sum_{\ell=1}^{2}\frac{\partial f\big(\epsilon_\ell(\mathbf{k})\big)}{\partial\,\epsilon_\ell(\mathbf{k})}\;\boldsymbol{\nabla}_\mathbf{k}\,\epsilon_\ell(\mathbf{k})\cdot\boldsymbol{\nabla}_\mathbf{k}\,\epsilon_\ell(\mathbf{k}) = \frac{e^2}{2V}\sum_{\mathbf{k}}\sum_{\ell=1}^{2} f\big(\epsilon_\ell(\mathbf{k})\big)\,\nabla^2_\mathbf{k}\,\epsilon_\ell(\mathbf{k})\,, \tag{20}$$

is the bare value, and F^1 is again a weighted average over the Fermi surfaces of $F^1_0(\mathbf{k})$ and $F^1_3(\mathbf{k})$, see (16). In this case, $F^1/2$ is the correction to the optical mass with respect to the quasiparticle effective one, defined through the last equation in (20).

We emphasise that the Fermi liquid corrections to both anomalous Hall conductivity (18) and Drude weight (19) stem from the fact that we are studying the response to a uniform electric field, which has to be evaluated in the ω-limit. As a consequence, the dressed current vertex that enters the response functions is $\boldsymbol{J}^\omega$ instead of $\boldsymbol{J}^q$, the latter being the one obtainable from the Ward-Takahashi identity. In a Fermi liquid, $\boldsymbol{J}^\omega \neq \boldsymbol{J}^q$ because the quasiparticle Fermi surface induces a non-analyticity at $\omega = q = 0$, which is the key to the microscopic derivation of Landau's Fermi liquid theory [17]. It is tantalising to interpret the correction $\boldsymbol{J}^\omega - \boldsymbol{J}^q$ as due to an effective dipole moment carried by the quasiparticles, as proposed in [9], even though we do not have a clear physical argument in support.

To conclude, we mention that, if instead of a fully spin-polarised BHZ model as in (11), we considered the same model without explicitly breaking time-reversal symmetry, or models sharing similar topological ingredients, we could still discuss topological properties as the non-quantised intrinsic component of the spin Hall conductivity, see, e.g., [22, 23].

4 Conclusions

Landau's theory of topological Fermi liquids predicts that the residual interactions among the quasiparticles, the f-parameters, yield corrections not only to conventional thermodynamic susceptibilities and longitudinal transport coefficients, like the Drude weight, but also to the intrinsic anomalous Hall conductivity. The latter is therefore not expressible solely in terms of the Berry phases acquired by the quasiparticle Bloch states adiabatically evolving on the Fermi surface, as one would intuitively argue [1], and of the bare Chern number of occupied bands. Both contributions to the anomalous Hall conductivity are in fact renormalized by the interaction with the quasiparticles at the chemical potential, simply reflecting the well known

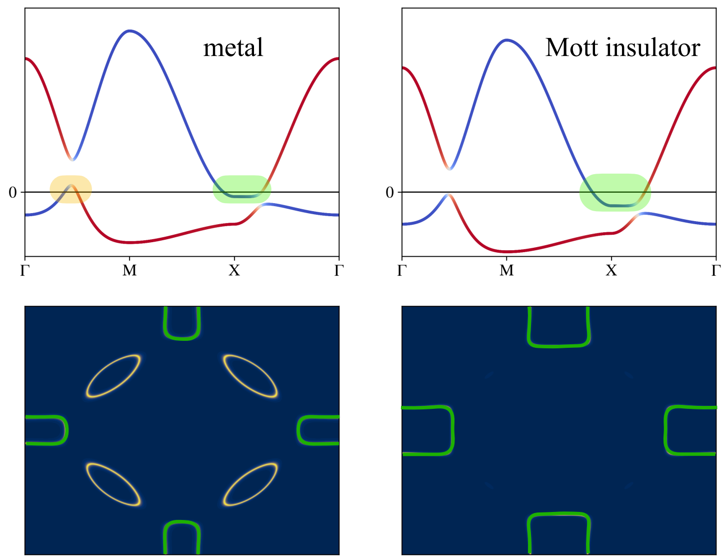

Figure 2: Left panels: hypothetical quasiparticle band structure in the metal phase (top) showing two distinct sets of quasiparticle Fermi pockets (bottom), one representing physical Fermi pockets, in yellow, which account for the hole doping, and the other, in green, being Luttinger pockets. Right panels: the same as left ones but on the Mott insulating side at $\delta = 0$. Here, only the Luttinger pockets exist.

difference between the static and dynamic limits of linear response functions in metals. This result, though not unexpected, has required a substantial extension of the original Noziéres and Luttinger microscopic derivation of conventional Fermi liquids [17], which we present in Appendix C.

The corrections to the anomalous Hall conductivity that we uncover is the metallic counterpart of recent results [10, 11] showing that the Chern number of two-dimensional quantum anomalous Hall insulators not necessarily coincides with the topological invariant [24–27] corresponding to the winding number $W(G)$, also denoted as $N_3(G)$, of the map $(\epsilon, \mathbf{k}) \to \hat{G}(i\epsilon, \mathbf{k}) \in \mathrm{GL}(n, \mathbb{C})$, where $\hat{G}(i\epsilon, \mathbf{k})$ is the fully-interacting Green's function. Indeed, as demonstrated in [10], this winding number is equivalent to that of the map $(\epsilon, \mathbf{k}) \to \hat{G}_*(i\epsilon, \mathbf{k})$, where $\hat{G}_*(i\epsilon, \mathbf{k})$ is obtained by filtering out from $\hat{G}(i\epsilon, \mathbf{k})$ the quasiparticle residue, see Appendix B, and thus coincides with the quasiparticle Green's function (3) in the doped insulator. Correspondingly, $W(G_*)$ reduces upon doping to the non-quantised σ_0^{H} in units of $e^2/2\pi$, and the corrections predicted in [10, 11] to the Fermi liquid ones we have just derived. We remark that, since $\hat{G}_*(i\epsilon, \mathbf{k})$ in the doped insulator has singular eigenvalues at $\epsilon = 0$ on the quasiparticle Fermi surface, $W(G_*)$ is, strictly speaking, not anymore a genuine winding number, though it remains perfectly defined. With this proviso, we hereafter refer to $W(G_*)$ still as the winding number.

In the following, we show that, elaborating on the previous observation, one can draw interesting and rather general conclusions. We start emphasising that Refs. [10] and [11] rationalise the puzzling evidence of Mott insulators with, presumably,[1] vanishing Chern number and yet finite winding number due to the existence of in-gap topological bands of Green's function zeros [28–33]. It is reasonable to conjecture that doping such Mott insulators with $\sigma^{\mathrm{H}} = 0$ but $W(G_*) \propto \sigma_0^{\mathrm{H}} \neq 0$ may lead, without any change of symmetry, to topological metals with

[1]The caution is due to the fact that in most of the cases we mention a direct calculation of the anomalous Hall conductivity has not been performed.

finite $\sigma^{\rm H}$. For simplicity, we assume that such metal can be described as in Sect. 3, and use the results of that section to infer how it may continuously transform into the Mott insulator as the doping $\delta \to 0$.

The first and most obvious temptation is to assume that $\delta \to 0$ simply corresponds to a quasiparticle Fermi surface that shrinks into a point and disappears in the Mott insulator. In that case, since the bare $\sigma^{\rm H}_0$ is expected to reach its quantised maximum magnitude when the valence band is full and the conduction one empty, the only possibility for $\sigma^{\rm H}$ in (18) to vanish is that $F^1_2 \to -2$ for $\delta \to 0$. However, from the definition of $F^1_2(\mathbf{k})$ in (16), we would rather conclude that $F^1_2 \to 0$ for $\delta \to 0$, unless $f^1_{2a}(\mathbf{k},\mathbf{k}')$, $a = 0,3$, diverge sufficiently fast to compensate the vanishing Fermi volume. As we earlier discussed, such possibility has to be discarded, which implies that the assumption of a quasiparticle Fermi surface that disappears when $\delta \to 0$ is not consistent with $\sigma^{\rm H} \to 0$, hence that a quasiparticle Fermi surface does survive till $\delta = 0$ to enforce $F^1_2 \to -2$. Since quasiparticle Fermi surfaces, poles of $\det\big(\hat{G}_*(0,\mathbf{k})\big)$, comprise both physical Fermi and Luttinger surfaces [34,35], poles and zeros of $\det\big(\hat{G}(0,\mathbf{k})\big)$, respectively, see also Appendix B, we must conclude that the quasiparticle Fermi surface that exists at the transition into the Mott insulator is actually a Luttinger surface, which is allowed also in Mott insulators. This, in turn, implies that one of the in-gap bands of zeros of the Mott insulator right after the transition must cross the chemical potential and thus form a Luttinger surface, which is the smooth evolution of that one in the weakly doped metal phase. In Fig. 2 we show a hypothetical quasiparticle band structure that realises the above scenario, obtained by properly tuning the parameters in (11). The metal phase, left panels, exhibits both Fermi pockets, in yellow, which account just for the hole doping [36], and Luttinger ones, in green. Only the latter remains in the Mott insulator at $\delta = 0$, right panels.

This physical scenario, which we have unveiled by quite general arguments, is fully consistent with the results of [10, 11], since it predicts a Luttinger surface in the Mott insulator which is known to yield a violation of Luttinger's theorem [36, 37]. Such violation directly explains, by means of the Streda formula [38, 39], why in the insulator $\sigma^{\rm H}$ can vanish despite $W(G) \neq 0$. We also mention that recent numerical simulations of model interacting topological insulators [33, 40] find evidence of bands of Green's function zeros in the topological phase before the Mott transition, and which, should they cross the chemical potential and form a Luttinger surface, could provide a simple explanation [40] for the intriguing properties of SmB_6 and YbB_{12} topological Kondo insulators.

If we take for granted the existence of a Luttinger surface in the topological metal at $\delta \ll 1$, we can draw a further conclusion. When $\delta \to 0$, the Drude weight (19) must vanish. However, D_0 in (20) is finite for $\delta \to 0$ because of the Luttinger surface, which implies that $D \to 0$ because $F^1 \to -2$. In other words, while the quasiparticle effective mass, defined by the last equality in (20), is smooth approaching the Mott phase, the optical mass diverges. Remarkably, this is precisely the scenario that might occur in a gapless quantum spin liquid [35], see also [41] and references therein, as well as the one that Grilli and Kotliar uncovered [42] in the $t-J$ model by a large-N expansion around the saddle point within the slave-boson formalism. The analogy suggests that also in the weakly doped $t-J$ model there must be a Luttinger surface, and, possibly, also in other weakly doped Mott insulators, irrespective whether topology is involved. The advantage of the latter is that it allows straight reaching this conclusion because the winding number, a topological property of the Green's function, can well be finite in both metal and insulator.

Acknowledgments

We are very grateful to Raffaele Resta for useful discussions and comments.

A Precise definition of the Fermi-liquid response functions

The correspondence between the low frequency, long wavelength and low temperature physical response functions and those obtained from the Hamiltonian (1) treated by HF+RPA is in reality not straight. Indeed, that correspondence can be drawn only in the limits where one can make use of the Ward-Takahashi identities [17], therefore only in connection with densities associated to conserved quantities and their corresponding currents, and either in the q-limit or in the ω-limit depending on the specific response function.

This point is particularly important in multiband models and for the current-current response function. Let us therefore begin by discussing the latter [17, 19]. The Bethe-Salpeter equation for the reducible vertex in terms of the f-parameters is

$$\Gamma_{\ell m,np}(\mathbf{k},\mathbf{k}';i\omega,\mathbf{q}) = f_{\ell m,np}(\mathbf{k},\mathbf{k}') + \frac{1}{V}\sum_{\mathbf{p}}\sum_{qs} f_{\ell s,qp}(\mathbf{k},\mathbf{p})\, R_{sq}(\mathbf{p};i\omega,\mathbf{q})\, \Gamma_{qm,ns}(\mathbf{p},\mathbf{k}';i\omega,\mathbf{q}), \quad \text{(A.1)}$$

where ω and $\mathbf{q}$ are, respectively, the frequency and momentum transferred in the particle-hole channel, and the kernel

$$\begin{aligned} R_{sq}(\mathbf{p};i\omega,\mathbf{q}) &= T\sum_{\epsilon} \frac{1}{i\epsilon + i\omega - \epsilon_q(\mathbf{p}+\mathbf{q})}\,\frac{1}{i\epsilon - \epsilon_s(\mathbf{p})} \\ &= \frac{f\big(\epsilon_s(\mathbf{p})\big) - f\big(\epsilon_q(\mathbf{p}+\mathbf{q})\big)}{i\omega + \epsilon_s(\mathbf{p}) - \epsilon_q(\mathbf{p}+\mathbf{q})}\,. \end{aligned} \quad \text{(A.2)}$$

Equation (A.1) can be shortly written as

$$\Gamma = f + f\odot R\odot\Gamma = f + \Gamma\odot R\odot f\,, \quad \text{(A.3)}$$

where the symbol $\odot$ denotes sum of internal indices and momenta. Hereafter, we make a series of formal and exact manipulations of the Bethe-Salpeter equation similar to those exploited by Noziéres and Luttinger [17] to derive Landau's Fermi liquid theory. We decided to show them explicitly since they might be not familiar to everybody.

From (A.3) we get

$$f = \Gamma\odot\big(1 + R\odot\Gamma\big)^{-1}. \quad \text{(A.4)}$$

We can take the q-limit of (A.3), sending first $\omega\to 0$ and then $\mathbf{q}\to\mathbf{0}$,

$$\Gamma^q = f + \Gamma^q\odot R^q\odot f = f + f\odot R^q\odot\Gamma\,, \quad \text{(A.5)}$$

and find

$$f = \big(1 + \Gamma^q\odot R^q\big)^{-1}\odot\Gamma^q\,. \quad \text{(A.6)}$$

Comparing (A.4) with (A.6) we obtain

$$\Gamma\big(1 + R\odot\Gamma\big)^{-1} = \big(1 + \Gamma^q\odot R^q\big)^{-1}\,\Gamma^q\,,$$

which implies

$$\big(1 + \Gamma^q\odot R^q\big)\Gamma = \Gamma^q\big(1 + R\odot\Gamma\big),$$

hence

$$\Gamma = \Gamma^q + \Gamma^q\odot\big(R - R^q\big)\odot\Gamma = \Gamma^q + \Gamma\odot\big(R - R^q\big)\odot\Gamma^q\,. \quad \text{(A.7)}$$

On the other hand, from (A.7) it follows that

$$\Gamma^q = \Gamma\odot\Big[1 + \big(R - R^q\big)\odot\Gamma\Big]^{-1},$$

and so,

$$\begin{aligned} 1+R^q\odot\Gamma^q &= 1+R^q\odot\Gamma\odot\Big[1+\big(R-R^q\big)\odot\Gamma\Big]^{-1} \\ &= \Big[1+\big(R-R^q\big)\odot\Gamma\Big]\odot\Big[1+\big(R-R^q\big)\odot\Gamma\Big]^{-1} \\ &\quad +R^q\odot\Gamma\odot\Big[1+\big(R-R^q\big)\odot\Gamma\Big]^{-1} \\ &= \big(1+R\odot\Gamma\big)\odot\Big[1+\big(R-R^q\big)\odot\Gamma\Big]^{-1}. \end{aligned} \tag{A.8}$$

The dressed current vertex $\boldsymbol{J}$ is obtained from the bare one $\boldsymbol{J}_0$, which we actually do not know, through the Bethe-Salpeter equation

$$\boldsymbol{J}=\boldsymbol{J}_0+\boldsymbol{J}_0\odot R\odot\Gamma=\boldsymbol{J}_0+\Gamma\odot R\odot\boldsymbol{J}_0\,, \tag{A.9}$$

whose q-limit is $\boldsymbol{J}^q=\boldsymbol{J}_0+\boldsymbol{J}_0\odot R^q\odot\Gamma^q$. As before, we can use (A.9) to solve for the unknown $\boldsymbol{J}_0$,

$$\boldsymbol{J}_0=\boldsymbol{J}\odot\big(1+R\odot\Gamma\big)^{-1}, \tag{A.10}$$

and, taking the q-limit as well as making use of (A.8),

$$\begin{aligned} \boldsymbol{J}_0 &= \boldsymbol{J}^q\odot\big(1+R^q\odot\Gamma^q\big)^{-1} \\ &= \boldsymbol{J}^q\odot\Big[1+\big(R-R^q\big)\odot\Gamma\Big]\odot\big(1+R\odot\Gamma\big)^{-1} \\ &= \big(1+\Gamma\odot R\big)^{-1}\odot\Big[1+\Gamma\odot\big(R-R^q\big)\Big]\odot\boldsymbol{J}^q\,, \end{aligned} \tag{A.11}$$

which, compared with (A.10), leads to

$$\boldsymbol{J}=\boldsymbol{J}^q+\boldsymbol{J}^q\odot\big(R-R^q\big)\odot\Gamma=\boldsymbol{J}^q+\Gamma\odot\big(R-R^q\big)\odot\boldsymbol{J}^q\,. \tag{A.12}$$

The current-current response function is defined, still in short notations, through

$$\chi_{JJ}\equiv\mathrm{Tr}\Big(\boldsymbol{J}\odot R\odot\boldsymbol{J}_0\Big)=\mathrm{Tr}\Big(\boldsymbol{J}_0\odot R\odot\boldsymbol{J}\Big), \tag{A.13}$$

whose q-limit is therefore $\chi^q=\mathrm{Tr}\Big(\boldsymbol{J}^q\odot R^q\odot\boldsymbol{J}_0\Big)$. We can thus write, making use of (A.9) and (A.12),

$$\begin{aligned} \chi_{JJ} &= \chi^q+\mathrm{Tr}\Big(\boldsymbol{J}\odot R\odot\boldsymbol{J}_0\Big)-\mathrm{Tr}\Big(\boldsymbol{J}^q\odot R^q\odot\boldsymbol{J}_0\Big) \\ &= \chi^q+\mathrm{Tr}\Big(\boldsymbol{J}^q\odot\big(R-R^q\big)\odot\boldsymbol{J}_0\Big)+\mathrm{Tr}\Big(\boldsymbol{J}^q\odot\big(R-R^q\big)\odot\Gamma\odot R\odot\boldsymbol{J}_0\Big) \\ &= \chi^q+\mathrm{Tr}\Big(\boldsymbol{J}^q\odot\Delta\odot\boldsymbol{J}\Big) \\ &= \chi^q+\mathrm{Tr}\Big(\boldsymbol{J}^q\odot\Delta\odot\boldsymbol{J}^q\Big)+\mathrm{Tr}\Big(\boldsymbol{J}^q\odot\Delta\odot\Gamma\odot\Delta\odot\boldsymbol{J}^q\Big), \end{aligned} \tag{A.14}$$

where we have defined $\Delta=R-R^q$. We can now finally make use of symmetries and of the Ward-Takahashi identities. We observe that gauge symmetry entails that χ^q must precisely cancel the diamagnetic term, which we also do not know, and the Ward-Takahashi identities that $\boldsymbol{J}^q(\mathbf{k})\equiv\boldsymbol{\nabla}_{\mathbf{k}}\hat{H}_*(\mathbf{k})$ [17]. Correspondingly, the component $\sigma_{ab}(i\omega,\mathbf{q})$ of the conductivity tensor reads

$$\sigma_{ab}=i\,e^2\lim_{\omega\to0}\lim_{\mathbf{q}\to\mathbf{0}}\frac{1}{i\omega}\left\{\mathrm{Tr}\Big(J^q_a\odot\Delta\odot J^q_b\Big)+\mathrm{Tr}\Big(J^q_a\odot\Delta\odot\Gamma\odot\Delta\odot J^q_b\Big)\right\}, \tag{A.15}$$

thus involving the ω-limit, first $\mathbf{q}\to\mathbf{0}$ and then $\omega\to0$.

It is worth noticing that in (A.14) the kernel R is substituted by $\Delta=R-R^q$. In other words, we have been obliged to manipulate RPA in order to build a correspondence with the physical response function and get rid of the unknown bare current vertex and diamagnetic term. From the explicit expression of R in (A.2), we find that

$$\begin{aligned}\Delta_{\ell m}(\mathbf{k};\omega,\mathbf{q}) &= R_{\ell m}(\mathbf{k};\omega,\mathbf{q}) - R^q_{\ell m}(\mathbf{k}) \\ &= \frac{f\big(\epsilon_m(\mathbf{k})\big)-f\big(\epsilon_\ell(\mathbf{k}+\mathbf{q})\big)}{i\omega+\big(\epsilon_m(\mathbf{k})-\epsilon_\ell(\mathbf{k}+\mathbf{q})\big)} - \lim_{\mathbf{q}\to 0}\frac{f\big(\epsilon_m(\mathbf{k})\big)-f\big(\epsilon_\ell(\mathbf{k}+\mathbf{q})\big)}{\epsilon_m(\mathbf{k})-\epsilon_\ell(\mathbf{k}+\mathbf{q})}\,.\end{aligned} \tag{A.16}$$

Recalling that we are in any case interested in small $\mathbf{q}$ and ω, then (A.16) for $\ell = m$ is at leading order, and defining $\boldsymbol{v}_\ell(\mathbf{k}) = \boldsymbol{\nabla}_\mathbf{k}\,\epsilon_\ell(\mathbf{k})$ the group velocity,

$$\Delta_{\ell\ell}(\mathbf{k};\omega,\mathbf{q}) \simeq -\frac{\partial f\big(\epsilon_\ell(\mathbf{k})\big)}{\partial\epsilon_\ell(\mathbf{k})}\,\frac{i\omega}{i\omega-\boldsymbol{v}_\ell(\mathbf{k})\cdot\mathbf{q}}\,, \tag{A.17}$$

which is the standard result in the single-band case [17]. On the contrary, for $\ell \neq m$ and assuming $\epsilon_\ell(\mathbf{k}) \neq \epsilon_m(\mathbf{k})$, we readily find that

$$\Delta_{\ell m}(\mathbf{k};\omega,\mathbf{q}) \simeq -i\omega\,\frac{f\big(\epsilon_m(\mathbf{k})\big)-f\big(\epsilon_\ell(\mathbf{k})\big)}{\big(\epsilon_\ell(\mathbf{k})-\epsilon_m(\mathbf{k})\big)^2}\,, \tag{A.18}$$

which vanishes linearly in ω. Since the ω-limit appears in the conductivity, then

$$\begin{aligned}\Delta^\omega_{\ell m}(\mathbf{k}) &= \lim_{\omega\to 0}\lim_{\mathbf{q}\to\mathbf{0}}\Delta_{\ell m}(\mathbf{k},\omega,\mathbf{q}) = -\delta_{m\ell}\,\frac{\partial f\big(\epsilon_\ell(\mathbf{k})\big)}{\partial\epsilon_\ell(\mathbf{k})}\,,\\ \dot{\Delta}^\omega_{\ell m}(\mathbf{k}) &= \lim_{\omega\to 0}\lim_{\mathbf{q}\to\mathbf{0}}\frac{\partial\Delta_{\ell m}(\mathbf{k},\omega,\mathbf{q})}{\partial i\omega} = -\big(1-\delta_{m\ell}\big)\,\frac{f\big(\epsilon_m(\mathbf{k})\big)-f\big(\epsilon_\ell(\mathbf{k})\big)}{\big(\epsilon_\ell(\mathbf{k})-\epsilon_m(\mathbf{k})\big)^2}\,.\end{aligned} \tag{A.19}$$

In other words, the matrix Δ^ω is diagonal while its derivative $\dot{\Delta}^\omega$ with respect to ω and calculated at $\omega = 0$ is off-diagonal. This result is important in the calculation of the Hall conductivity.

However, let us at first calculate the longitudinal conductivity, i.e., the variation of the electric current in the same direction, e.g., x, of a uniform and static electric field, divided by the field strength. From (A.15), moving to the real frequency axis, $i\omega \to \omega + i0^+$ with small ω and setting $\mathbf{q} = \mathbf{0}$, we find

$$\begin{aligned}\sigma_{xx}(\omega) &= i\,\frac{e^2}{\omega+i0^+}\left\{\mathrm{Tr}\Big(J^q_x\odot\Delta^\omega\odot J^q_x\Big)+\mathrm{Tr}\Big(J^q_x\odot\Delta^\omega\odot\Gamma^\omega\odot\Delta^\omega\odot J^q_x\Big)\right\}\\ &= i\,\frac{1}{\omega+i0^+}\,e^2\,\mathrm{Tr}\Big(J^q_x\odot\Delta^\omega\odot J^\omega_x\Big) \equiv i\,\frac{1}{\omega+i0^+}\,D_{xx}\,,\end{aligned} \tag{A.20}$$

where D_{xx} is the Drude weight and $\boldsymbol{J}^\omega$ the ω-limit of the current vertex that, through (A.12), satisfies

$$\boldsymbol{J}^\omega = \boldsymbol{J}^q + \boldsymbol{J}^q\odot\Delta^\omega\odot\Gamma^\omega = \boldsymbol{J}^q + \Gamma^\omega\odot\Delta^\omega\odot\boldsymbol{J}^q\,. \tag{A.21}$$

Let us now calculate the Hall conductivity through the matrix elements σ_{ab} of (A.15). For that we can simply adapt the results in [10], which imply that the antisymmetric combination $(\sigma_{ab}-\sigma_{ba})/2$ defines the element σ^{H}_{ab} of the anomalous Hall conductivity, which can be shown are simply

$$\sigma^{\mathrm{H}}_{ab} = i\,e^2\lim_{\omega\to 0}\lim_{\mathbf{q}\to\mathbf{0}}\frac{\partial}{\partial i\omega}\left\{\mathrm{Tr}\Big(J^q_a\odot\Delta\odot J^q_b\Big)+\mathrm{Tr}\Big(J^q_a\odot\Delta\odot\Gamma\odot\Delta\odot J^q_b\Big)\right\}. \tag{A.22}$$

Therefore,

$$\begin{aligned}\sigma^{\mathrm{H}}_{ab} = i\,e^2 \Big\{ &\mathrm{Tr}\Big(J^q_a \odot \dot{\Delta}^\omega \odot J^q_b\Big) + \mathrm{Tr}\Big(J^q_a \odot \dot{\Delta}^\omega \odot \Gamma^\omega \odot \Delta^\omega \odot J^q_b\Big) \\ &+ \mathrm{Tr}\Big(J^q_a \odot \Delta^\omega \odot \Gamma^\omega \odot \dot{\Delta}^\omega \odot J^q_b\Big) + \mathrm{Tr}\Big(J^q_a \odot \Delta^\omega \odot \dot{\Gamma}^\omega \odot \Delta^\omega \odot J^q_b\Big)\Big\}\,,\end{aligned} \tag{A.23}$$

involving Δ^ω and $\dot{\Delta}^\omega$ in (A.19). We note that the ω-limit of the derivative of the Bethe-Salpeter equation (A.3) is

$$\dot{\Gamma}^\omega = f \odot \dot{\Delta}^\omega \odot \Gamma^\omega + f \odot R^\omega \odot \dot{\Gamma}^\omega\,,$$

which leads to

$$\dot{\Gamma}^\omega = \Gamma^\omega\,\dot{\Delta}^\omega\,\Gamma^\omega\,,$$

and, substituted into (A.23), to the compact expression

$$\sigma^{\mathrm{H}}_{ab} = i\,e^2\,\mathrm{Tr}\Big(J^\omega_a \odot \dot{\Delta}^\omega \odot J^\omega_b\Big)\,, \tag{A.24}$$

which involves again the ω-limit of the current vertex (A.21).

Let us finally consider the charge density-density response function. We can repeat step-by-step the above manipulations but now using the ω-limit of the vertex. The final result is that the charge density-density response function reads

$$\chi_{\rho\rho} = \chi^\omega_{\rho\rho} + \mathrm{Tr}\Big(\rho^\omega \odot \big(R - R^\omega\big) \odot \rho^\omega\Big) + \mathrm{Tr}\Big(\rho^\omega \odot \big(R - R^\omega\big) \odot \Gamma \odot \big(R - R^\omega\big) \odot \rho^\omega\Big)\,, \tag{A.25}$$

where ρ^ω is the ω-limit of the charge density vertex,

$$\chi^\omega_{\rho\rho} = \mathrm{Tr}\Big(\rho^\omega \odot R^\omega \odot \rho_0\Big)\,, \tag{A.26}$$

the ω-limit of the response function, and ρ_0 the bare vertex. Charge conservation implies that $\chi^\omega_{\rho\rho} = 0$, while the Ward-Takahashi identities that ρ^ω is the identity matrix in orbital space [17]. Since $R^q - R^\omega = -\Delta^\omega$, see (A.19), the charge compressibility is readily obtained as

$$\begin{aligned}\kappa = -\chi^q_{\rho\rho} &= \mathrm{Tr}\Big(\Delta^\omega\Big) - \mathrm{Tr}\Big(\Delta^\omega \odot \Gamma^q \odot \Delta^\omega\Big) \\ &= -\frac{1}{V}\sum_{\mathbf{k}\ell} \frac{\partial f\big(\epsilon_\ell(\mathbf{k})\big)}{\partial \epsilon_\ell(\mathbf{k})} - \frac{1}{V^2}\sum_{\mathbf{k}\mathbf{k}'\ell\ell'} \frac{\partial f\big(\epsilon_\ell(\mathbf{k})\big)}{\partial \epsilon_\ell(\mathbf{k})}\,\Gamma^q_{\ell\ell',\ell'\ell}(\mathbf{k},\mathbf{k}')\,\frac{\partial f\big(\epsilon_{\ell'}(\mathbf{k}')\big)}{\partial \epsilon_{\ell'}(\mathbf{k}')}\,,\end{aligned} \tag{A.27}$$

which is the conventional result of Fermi liquid theory showing that the compressibility is not just the quasiparticle density-of-states at the chemical potential, but acquires a correction from the quasiparticle interaction.

B Quasiparticle Green's function

In this Appendix we show how to rigorously define the quasiparticle Green's function $\hat{G}_*(i\epsilon, \mathbf{k})$ through the fully interacting thermal one $\hat{G}(i\epsilon, \mathbf{k})$ of the physical electrons, having in mind a multiband system in which both Green's functions are matrices. This will give us the opportunity to clarify some results mentioned in the main text.

The physical Green's function $\hat{G}(i\epsilon, \mathbf{k})$ satisfies the Dyson equation

$$\hat{G}(i\epsilon, \mathbf{k})^{-1} = i\epsilon - \hat{H}_0(\mathbf{k}) - \hat{\Sigma}(i\epsilon, \mathbf{k})\,,$$

where $\hat{H}_0(\mathbf{k})$ is the non-interacting Hamiltonian, not to be confused with the quasiparticle one in (1), and the self-energy $\hat{\Sigma}(i\epsilon,\mathbf{k})$ accounts for all interaction effects. We recall that $\hat{\Sigma}(i\epsilon,\mathbf{k})^\dagger = \hat{\Sigma}(-i\epsilon,\mathbf{k})$, which implies that

$$\hat{\Sigma}_1(i\epsilon,\mathbf{k}) = \frac{1}{2}\Big(\hat{\Sigma}(i\epsilon,\mathbf{k}) + \hat{\Sigma}(i\epsilon,\mathbf{k})^\dagger\Big) = \frac{1}{2}\Big(\hat{\Sigma}(i\epsilon,\mathbf{k}) + \hat{\Sigma}(-i\epsilon,\mathbf{k})\Big),$$

is hermitean and even in ϵ, while

$$\hat{\Sigma}_2(i\epsilon,\mathbf{k}) = \frac{1}{2i}\Big(\hat{\Sigma}(i\epsilon,\mathbf{k}) - \hat{\Sigma}(i\epsilon,\mathbf{k})^\dagger\Big) = \frac{1}{2i}\Big(\hat{\Sigma}(i\epsilon,\mathbf{k}) - \hat{\Sigma}(-i\epsilon,\mathbf{k})\Big),$$

is still hermitean but odd in ϵ. In addition, its eigenvalues are negative for positive ϵ, positive for negative ϵ, and vanish when ϵ is strictly zero.

One defines [35] a semi-positive definite matrix

$$\hat{Z}(\epsilon,\mathbf{k}) = \left(1 - \frac{\hat{\Sigma}_2(i\epsilon,\mathbf{k})}{\epsilon}\right)^{-1} = \hat{A}(\epsilon,\mathbf{k})^\dagger \hat{A}(\epsilon,\mathbf{k}), \tag{B.1}$$

with eigenvalues $\in [0,1]$, which plays the role of the quasiparticle residue, and the hermitian frequency-dependent Hamiltonian

$$\hat{H}_*(\epsilon,\mathbf{k}) = \hat{H}_*(-\epsilon,\mathbf{k}) = \hat{A}(\epsilon,\mathbf{k})\Big(\hat{H}_0(\mathbf{k}) + \hat{\Sigma}_1(i\epsilon,\mathbf{k})\Big)\hat{A}(\epsilon,\mathbf{k})^\dagger, \tag{B.2}$$

through which

$$\hat{G}(i\epsilon,\mathbf{k}) = \hat{A}(\epsilon,\mathbf{k})^\dagger \, \frac{1}{i\epsilon - \hat{H}_*(\epsilon,\mathbf{k})} \, \hat{A}(\epsilon,\mathbf{k}). \tag{B.3}$$

The quasiparticle Green's function (3) is simply obtained through the low-frequency limit of $\big(i\epsilon - \hat{H}_*(\epsilon,\mathbf{k})\big)^{-1}$ in (B.3), which, provided $\hat{H}_*(\epsilon,\mathbf{k}) = \hat{H}_*(-\epsilon,\mathbf{k})$ has a regular Taylor expansion in ϵ, i.e., $\hat{H}_*(\epsilon,\mathbf{k}) \simeq \hat{H}_*(0,\mathbf{k}) + O\big(\epsilon^2\big)$, reads, at leading order,

$$\hat{G}_*(i\epsilon,\mathbf{k}) \simeq \frac{1}{i\epsilon - \hat{H}_*(0,\mathbf{k})} \equiv \frac{1}{i\epsilon - \hat{H}_*(\mathbf{k})}. \tag{B.4}$$

The *quasiparticle Fermi surface* corresponds to the manifold $\mathbf{k} = \mathbf{k}_{*F}$ in momentum space where

$$\det\Big(\hat{H}_*(\mathbf{k}_{*F})\Big) = \det\Big(\hat{Z}(0,\mathbf{k}_{*F})\Big)\det\Big(\hat{H}_0(\mathbf{k}_{*F}) + \hat{\Sigma}_1(0,\mathbf{k}_{*F})\Big) = 0, \tag{B.5}$$

which includes the roots of both terms on the right hand side, and where $\hat{Z}(0,\mathbf{k})$ must be evaluated through the limit $\epsilon \to 0$ of (B.1). If $\det\big(\hat{Z}(0,\mathbf{k})\big)$ is finite, we observe that, since

$$\hat{G}(0,\mathbf{k}) = -\frac{1}{\hat{H}_0(\mathbf{k}) + \hat{\Sigma}_1(0,\mathbf{k})},$$

the roots of $\det\big(\hat{H}_0(\mathbf{k}) + \hat{\Sigma}_1(0,\mathbf{k})\big)$ are actually the poles of $\det\big(\hat{G}(0,\mathbf{k})\big)$ and define the physical Fermi surface, $\mathbf{k} = \mathbf{k}_F$. On the contrary, one realises through (B.3) that the roots of $\det\big(\hat{Z}(0,\mathbf{k})\big) = \big|\det\big(\hat{A}(0,\mathbf{k})\big)\big|^2$ correspond to those of $\det\big(\hat{G}(0,\mathbf{k})\big)$, which thus define the physical Luttinger surface $\mathbf{k} = \mathbf{k}_L$. More precisely, $\det\big(\hat{\Sigma}_1(0,\mathbf{k})\big)$ has a simple pole on the Luttinger surface, thus $\det\big(\hat{G}(0,\mathbf{k}_L)\big) = 0$, while $\det\big(\hat{Z}(0,\mathbf{k})\big)$ a second order root, so that (B.5) is indeed verified for $\mathbf{k}_{*F} = \mathbf{k}_L$. It follows that the *quasiparticle Fermi surface* $\mathbf{k}_{*F} = \mathbf{k}_F \cup \mathbf{k}_L$ comprises both physical Fermi and Luttinger surfaces, as we mentioned in the main text.

In reality, (B.3) is an exact factorisation of the physical electron Green's function that remains valid also in insulators lacking Fermi and Luttinger surfaces, and which was exploited in [10] to explicitly calculate the winding number $W(G)$ in two dimensions. Indeed, since $W(M\,N) = W(M) + W(N)$ and $W(M) = 0$ if $M = M^\dagger$, then

$$W(G) = W(A^\dagger\, G_*\, A) = W(G_*) + W(A^\dagger A) = W(G_*)\,.$$

One can rigorously prove [10] that $W(G_*)$ reduces to the well-known TKNN formula [43] calculated with the eigenstates of $\hat{H}_*(\mathbf{k})$ in (B.4). The derivation remains simply true even if there is a quasiparticle Fermi surface, a result we used in the Sect. 4.

C Formal derivation of Landau's Fermi liquid theory

Assuming the case in which a quasiparticle Fermi surface exists, Landau's Fermi liquid theory can be microscopically derived [17, 19] under the sole condition that $\hat{Z}(\epsilon,\mathbf{k})$ and $\hat{H}_*(\epsilon,\mathbf{k})$ are analytic matrix-valued functions of ϵ in the vicinity of the origin $\epsilon = 0$ and of $\mathbf{k}$ close to the quasiparticle Fermi surface $\mathbf{k}_{*F}$. This condition is verified not only near a physical Fermi surface, $\mathbf{k} \simeq \mathbf{k}_F$, but also near a Luttinger surface [34, 35], $\mathbf{k} \simeq \mathbf{k}_L$, despite the singular self-energy. Moreover, we emphasise that the Nozières and Luttinger derivation [17] is fully non-perturbative, and, therefore, perfectly valid also when a Luttinger surface is present, which does entail a breakdown of perturbation theory.

Since most of the technical details have been already presented in Appendix A, we end by briefly sketching how Nozières and Luttinger derivation [17] works to appreciate its elegance and non-perturbative character. Moreover, we attempt a generalisation of the theory to a multi-band Fermi liquid to assess under which conditions the results in Appendix A do reproduce the physical thermodynamic susceptibilities and transport coefficients.

Let us therefore consider again the Bethe-Salpeter equations (A.3) and (A.9) for the scattering four-leg vertex Γ and the current, $\boldsymbol{J}$, or the density, ρ, vertices, now, however, written for the physical electrons. For convenience, we hereafter focus on the current-current response function, since the extension to the density-density one is straightforward. The Bethe-Salpeter equations involve a kernel R, which is a tensor that depends on four indices in the chosen basis of single-particle wavefunctions. Specifically,

$$\begin{aligned} R_{\alpha\beta,\gamma\delta}(i\epsilon,\mathbf{k};i\omega,\mathbf{q}) &= G_{\alpha\beta}(i\epsilon+i\omega,\mathbf{k}+\mathbf{q})\,G_{\gamma\delta}(i\epsilon,\mathbf{k}) \\ &= \left(\frac{1}{i\epsilon+i\omega-H_0(\mathbf{k}+\mathbf{q})-\Sigma(i\epsilon+i\omega,\mathbf{k}+\mathbf{q})}\right)_{\alpha\beta}\left(\frac{1}{i\epsilon-H_0(\mathbf{k})-\Sigma(i\epsilon,\mathbf{k})}\right)_{\gamma\delta}. \end{aligned} \tag{C.1}$$

We now use the exact representation of $\hat{G}(i\epsilon,\mathbf{k})$ in (B.3) and define the quasiparticle vertices Γ_* and $\boldsymbol{J}_*$ by contracting each incoming external leg of the physical vertices with $\hat{A}(\epsilon,\mathbf{k})$ and each outgoing one with $\hat{A}(\epsilon,\mathbf{k})^\dagger$. The result is that in the Bethe-Salpeter equations (A.3) and (A.9) all vertices are replaced by the quasiparticle ones, and the kernel (C.1) by

$$\begin{aligned} R_{*\,\alpha\beta,\gamma\delta}(i\epsilon,\mathbf{k};i\omega,\mathbf{q}) &= \mathcal{G}_{\alpha\beta}(i\epsilon+i\omega,\mathbf{k}+\mathbf{q})\,\mathcal{G}_{\gamma\delta}(i\epsilon,\mathbf{k}) \\ &= \left(\frac{1}{i\epsilon+i\omega-\hat{H}_*(\epsilon+\omega,\mathbf{k}+\mathbf{q})}\right)_{\alpha\beta}\left(\frac{1}{i\epsilon-\hat{H}_*(\epsilon,\mathbf{k})}\right)_{\gamma\delta}. \end{aligned} \tag{C.2}$$

Without making any assumption whatsoever, one can formally manipulate the new Bethe-Salpeter equations, as discussed in Appendix A, so as to express them in terms of the vertices Γ_*^q and $\boldsymbol{J}_*^q$ calculated in the static q-limit, where the latter is fully determined by the Ward-Takahashi identities [17]. The outcome of such exact manipulation is that the kernel R_* in (C.2) is replaced by $R_* - R_*^q$, which, following [17], we would like to represent in the sense of a distribution Δ_* in the Matsubara frequency ϵ. Let us discuss separately the derivation of Δ_* and of its derivative $\partial_{i\omega}\Delta_*$ at $\omega = 0$, starting from the former.

C.1 Kernel for the longitudinal conductivity

We note that $\hat{\mathcal{G}}(i\epsilon,\mathbf{k})$ is continuous for any $\epsilon\neq 0$, while at $\epsilon=0$ its anti-hermitian part changes sign discontinuously. It follows that R_* in (C.2) has two discontinuities at $\epsilon=0$ and $\epsilon=-\omega$, whereas R_*^q a single one at $\epsilon=0$. Since we are interested in small ω and $\mathbf{q}$, actually in their ω-limit for the calculation of transport coefficients, we realise that the only region in ϵ where R_* differs from R_*^q is when $-\omega<\epsilon<0$, assuming for simplicity $\omega>0$. Since ω is small and $\hat{H}_*(\epsilon,\mathbf{k})$ is even in ϵ, in this region we can approximate

$$\hat{\mathcal{G}}(i\epsilon,\mathbf{k})=\frac{1}{i\epsilon-\hat{H}_*(\epsilon,\mathbf{k})}\simeq\frac{1}{i\epsilon-\hat{H}_*(0,\mathbf{k})}=\frac{1}{i\epsilon-\hat{H}_*(\mathbf{k})}=\hat{G}_*(i\epsilon,\mathbf{k}), \tag{C.3}$$

provided $\hat{H}_*(\epsilon,\mathbf{k})$ is analytic around $\epsilon=0$. It follows that, if $F(i\epsilon)$ is a test function smooth around $\epsilon=0$ and we formally write $R_*=\hat{\mathcal{G}}(i\epsilon+i\omega,\mathbf{k}+\mathbf{q})\otimes\hat{\mathcal{G}}(i\epsilon,\mathbf{k})$ then

$$\begin{aligned}T\sum_\epsilon\big(R_*-R_*^q\big)F(i\epsilon)\xrightarrow[\omega\text{-limit}]{}&T\sum_{-\omega<\epsilon<0}\hat{\mathcal{G}}(i\epsilon+i\omega,\mathbf{k})\otimes\hat{\mathcal{G}}(i\epsilon,\mathbf{k})\,F(i\epsilon)\\&\simeq F(0)\,T\sum_{-\omega<\epsilon<0}\hat{G}_*(i\epsilon+i\omega,\mathbf{k})\otimes\hat{G}_*(i\epsilon,\mathbf{k})\,.\end{aligned}$$

Now, we can project onto the basis that diagonalises $\hat{H}_*(\mathbf{k})$, and, specifically, on the element (ℓ,m), so that

$$\begin{aligned}&F(0)\,T\sum_{-\omega<\epsilon<0}\hat{G}_*(i\epsilon+i\omega,\mathbf{k})\otimes\hat{G}_*(i\epsilon,\mathbf{k})\\&\to F(0)\,T\sum_{-\omega<\epsilon<0}\frac{1}{i\epsilon+i\omega-\epsilon_\ell(\mathbf{k})}\,\frac{1}{i\epsilon-\epsilon_m(\mathbf{k})}=F(0)\,\frac{1}{i\omega-\epsilon_\ell(\mathbf{k})+\epsilon_m(\mathbf{k})}\,T\\&\qquad\times\sum_{-\omega<\epsilon<0}\left(\frac{1}{i\epsilon-\epsilon_m(\mathbf{k})}-\frac{1}{i\epsilon+i\omega-\epsilon_\ell(\mathbf{k})}\right).\end{aligned}$$

Since the sum is proportional to ω, a non-zero result for $\omega\to 0$ requires $\ell=m$, excluding the possibility of crossing bands. In this case

$$F(0)\,\frac{1}{i\omega}\,T\sum_{-\omega<\epsilon<0}\left(\frac{1}{i\epsilon-\epsilon_\ell(\mathbf{k})}-\frac{1}{i\epsilon+i\omega-\epsilon_\ell(\mathbf{k})}\right)\xrightarrow[\omega\to 0]{}-F(0)\,\frac{\partial f\big(\epsilon_\ell(\mathbf{k})\big)}{\partial\epsilon_\ell(\mathbf{k})}\,,$$

where $f(x)$ is the Fermi distribution function, which derives from

$$\begin{aligned}T\sum_{-\omega<\epsilon<0}\left(\frac{1}{i\epsilon-x}-\frac{1}{i\epsilon+i\omega-x}\right)&=T\sum_{0<\epsilon<\omega}\left(\frac{1}{-i\epsilon-x}-\frac{1}{i\epsilon-x}\right)\\&=T\sum_{\epsilon>0}\left(\frac{1}{i\epsilon+i\omega-x}-\frac{1}{i\epsilon-x}+\frac{1}{-i\epsilon-x}-\frac{1}{-i\epsilon-i\omega-x}\right)\\&\simeq-i\omega\,T\sum_{\epsilon>0}\left(\frac{1}{(i\epsilon-x)^2}+\frac{1}{(-i\epsilon-x)^2}\right)\\&=-i\omega\,T\sum_\epsilon\frac{1}{(i\epsilon-x)^2}\\&=-i\omega\,\frac{\partial f(x)}{\partial x}\,.\end{aligned}$$

In conclusion, the distribution Δ_* in the ω-limit is simply

$$\Delta^\omega_{*\ell m}(i\epsilon,\mathbf{k})=-\delta_{\ell m}\,\frac{\delta_{\epsilon 0}}{T}\,\frac{\partial f\big(\epsilon_\ell(\mathbf{k})\big)}{\partial\epsilon_\ell(\mathbf{k})}\,, \tag{C.4}$$

which coincides with (A.19) if we identify the eigenvalues $\epsilon_\ell(\mathbf{k})$ with the Hartree-Fock ones of the Hamiltonian (1).

We recall that $\hat{\boldsymbol{J}}^q_*(\epsilon,\mathbf{k}) = \boldsymbol{\nabla}_\mathbf{k}\hat{H}_*(\epsilon,\mathbf{k})$ is hermitian and even in ϵ, and, because of (C.4),

$$\begin{aligned}\hat{\boldsymbol{J}}^\omega_*(\epsilon,\mathbf{k}) &= \boldsymbol{\nabla}_\mathbf{k}\hat{H}_*(\epsilon,\mathbf{k}) + \frac{1}{V}\sum_{\mathbf{k}'}\hat{\Gamma}^\omega_*(i\epsilon,\mathbf{k};0,\mathbf{k}')\odot\hat{\Delta}^\omega(\mathbf{k}')\odot\boldsymbol{\nabla}_{\mathbf{k}'}\hat{H}_*(\mathbf{k}')\\ &= \boldsymbol{\nabla}_\mathbf{k}\hat{H}_*(\epsilon,\mathbf{k}) + \delta\hat{\boldsymbol{J}}^\omega_*(\epsilon,\mathbf{k}),\\ \hat{\Gamma}^\omega_*(i\epsilon,\mathbf{k};0,\mathbf{k}') &= \hat{\Gamma}^q_*(i\epsilon,\mathbf{k};0,\mathbf{k}') + \frac{1}{V}\sum_{\mathbf{p}}\hat{\Gamma}^q_*(i\epsilon,\mathbf{k};0,\mathbf{p})\odot\hat{\Delta}^\omega(\mathbf{p})\odot\hat{\Gamma}^\omega_*(0,\mathbf{p};0,\mathbf{k}'),\end{aligned}\tag{C.5}$$

where $\hat{\Gamma}^\omega_*(0\mathbf{k},0\mathbf{k}')$ and $\hat{\Gamma}^q_*(0\mathbf{k},0\mathbf{k}')$ must be identified with the Landau f and A parameters, respectively. Therefore, $\hat{\Gamma}^{\omega/q}_*(\epsilon\mathbf{k},0\mathbf{k}')$ must be continuous across $\epsilon=0$, and thus

$$\hat{\boldsymbol{J}}^\omega_*(\epsilon\to 0^+,\mathbf{k}) = \hat{\boldsymbol{J}}^\omega_*(\epsilon\to 0^-,\mathbf{k}) = \hat{\boldsymbol{J}}^\omega_*(\mathbf{k}) = \hat{\boldsymbol{J}}^\omega_*(\mathbf{k})^\dagger\,.\tag{C.6}$$

We can elaborate on further. We note that $\hat{\boldsymbol{J}}^q_*(\epsilon,\mathbf{k})$ and $\hat{\Gamma}^q_*(i\epsilon,\mathbf{k};i\epsilon',\mathbf{k}')$ are related to each other by a Bethe-Salpeter equation. Since the outcome of this equation is $\hat{\boldsymbol{J}}^q_*(\epsilon,\mathbf{k})$, which is even in ϵ, it is plausible, as can be indeed verified in the lowest order skeleton expansion, that $\hat{\Gamma}^q_*(i\epsilon,\mathbf{k};0,\mathbf{k}')$, and thus, through (C.5), $\hat{\Gamma}^\omega_*(i\epsilon,\mathbf{k};0,\mathbf{k}')$ are also even in ϵ. We thus conclude that $\hat{\boldsymbol{J}}^\omega_*(\epsilon,\mathbf{k}) = \hat{\boldsymbol{J}}^\omega_*(-\epsilon,\mathbf{k}) = \hat{\boldsymbol{J}}^\omega_*(\epsilon,\mathbf{k})^\dagger$ is hermitian. More explicitly, we find, through (C.5) and (C.4), that

$$\begin{aligned}\Gamma^\omega_{*\ell n,nm}(i\epsilon,\mathbf{k};0,\mathbf{k}') &= \Gamma^q_{*\ell n,nm}(i\epsilon,\mathbf{k};0,\mathbf{k}') - \frac{1}{V}\sum_{r\mathbf{p}}\Gamma^q_{*\ell r,rm}(i\epsilon,\mathbf{k};0,\mathbf{p})\frac{\partial f\big(\epsilon_r(\mathbf{p})\big)}{\partial\epsilon_r(\mathbf{p})}\,\Gamma^\omega_{*rn,nr}(0,\mathbf{p};0,\mathbf{k}')\\ &= \Gamma^q_{*\ell n,nm}(i\epsilon,\mathbf{k};0,\mathbf{k}') - \frac{1}{V}\sum_{r\mathbf{p}}\Gamma^q_{*\ell r,rm}(i\epsilon,\mathbf{k};0,\mathbf{p})\frac{\partial f\big(\epsilon_r(\mathbf{p})\big)}{\partial\epsilon_r(\mathbf{p})}\,f_{r\mathbf{p},n\mathbf{k}'}\,,\end{aligned}\tag{C.7}$$

and thus

$$\begin{aligned}\delta\boldsymbol{J}^\omega_{*\ell m}(\epsilon,\mathbf{k}) = &-\frac{1}{V}\sum_{\mathbf{k}',n}\Gamma^q_{*\ell n,nm}(i\epsilon,\mathbf{k};0,\mathbf{k}')\,\frac{\partial f\big(\epsilon_n(\mathbf{k}')\big)}{\partial\mathbf{k}'}\\ &+\frac{1}{V^2}\sum_{\mathbf{k}'\mathbf{p}}\sum_{nr}\Gamma^q_{*\ell r,rm}(i\epsilon,\mathbf{k};0,\mathbf{p})\,\frac{\partial f\big(\epsilon_r(\mathbf{p})\big)}{\partial\epsilon_r(\mathbf{p})}\,\frac{\partial f\big(\epsilon_n(\mathbf{k}')\big)}{\partial\mathbf{k}'}\,f_{r\mathbf{p},n\mathbf{k}'}\,.\end{aligned}\tag{C.8}$$

C.2 Hall conductivity

Since the distribution $\partial_{i\omega}\Delta_*\equiv\partial_{i\omega}R_*$ is needed to obtain the expression of the Hall conductivity (A.24), we derive it by explicitly calculating the latter quantity closely following [10].

For convenience, we here define

$$R_* = \hat{\mathcal{G}}(i\epsilon+i\omega,\mathbf{k}+\mathbf{q})\otimes\hat{\mathcal{G}}(i\epsilon-i\omega,\mathbf{k})\,,$$

so that

$$\partial_{i\omega}\Delta_* = \frac{\partial R_*}{\partial 2i\omega}\xrightarrow[\omega\text{-limit}]{} -\frac{i}{2}\Big(\partial_\epsilon\hat{\mathcal{G}}(i\epsilon,\mathbf{k})\otimes\hat{\mathcal{G}}(i\epsilon,\mathbf{k}) - \hat{\mathcal{G}}(i\epsilon,\mathbf{k})\otimes\partial_\epsilon\hat{\mathcal{G}}(i\epsilon,\mathbf{k})\Big).\tag{C.9}$$

Therefore, dropping for simplicity the asterisks in the current vertices, the 1-2 component of the Hall conductivity (A.24) can be formally written as

$$\begin{aligned}\sigma^{\mathrm{H}}_{12} &= \frac{e^2}{2V}\sum_{\mathbf{k}} T\sum_{\epsilon}\epsilon_{\mu\nu}\,\mathrm{Tr}\Big(\hat{J}^{\omega}_{\mu}(\epsilon,\mathbf{k})\,\partial_\epsilon\,\hat{\mathcal{G}}(i\epsilon,\mathbf{k})\,\hat{J}^{\omega}_{\nu}(\epsilon,\mathbf{k})\,\hat{\mathcal{G}}(i\epsilon,\mathbf{k})\Big)\\ &= -\frac{e^2}{2V}\sum_{\mathbf{k}} T\sum_{\epsilon}\epsilon_{\mu\nu}\,\mathrm{Tr}\Big(\hat{J}^{\omega}_{\mu}(\epsilon,\mathbf{k})\,\hat{\mathcal{G}}(i\epsilon,\mathbf{k})\,\partial_\epsilon\,\hat{\mathcal{G}}(i\epsilon,\mathbf{k})^{-1}\,\hat{\mathcal{G}}(i\epsilon,\mathbf{k})\hat{J}^{\omega}_{\nu}(\epsilon,\mathbf{k})\,\hat{\mathcal{G}}(i\epsilon,\mathbf{k})\Big)\\ &= -\frac{e^2}{2V}\sum_{\mathbf{k}} T\sum_{\epsilon}\epsilon_{\mu\nu}\sum_{\ell m n}\mathcal{G}_\ell(i\epsilon,\mathbf{k})\mathcal{G}_m(i\epsilon,\mathbf{k})\mathcal{G}_n(i\epsilon,\mathbf{k})J^{\omega}_{\mu,\ell m}(\epsilon,\mathbf{k})\left(\partial_\epsilon\,\hat{\mathcal{G}}(i\epsilon,\mathbf{k})^{-1}\right)_{mn} J^{\omega}_{\nu,n\ell}(\epsilon,\mathbf{k}),\end{aligned}\tag{C.10}$$

where the antisymmetric tensor $\epsilon_{\mu\nu}$ equals 1 if $\mu=1$ and $\nu=2$, and we use the basis that diagonalizes $\hat{H}_*(\epsilon,\mathbf{k})$, i.e., $\hat{H}_*(\epsilon,\mathbf{k})\,|u_\ell(\epsilon,\mathbf{k})\rangle=\epsilon_\ell(\epsilon,\mathbf{k})\,|u_\ell(\epsilon,\mathbf{k})\rangle$. One can readily verify that, because $\hat{\mathcal{G}}(i\epsilon,\mathbf{k})^\dagger=\hat{\mathcal{G}}(-i\epsilon,\mathbf{k})$ and the current vertices are hermitian, (C.10) is correctly real.

Since the last term in (C.10) with all indices equal, $l=m=n$, vanishes, we can write

$$\begin{aligned}\sigma^{\mathrm{H}}_{12} &= \frac{e^2}{2V}\sum_{\mathbf{k}} T\sum_{\epsilon}\epsilon_{\mu\nu}\sum_{\ell\neq m}\partial_\epsilon\,\mathcal{G}_m(i\epsilon,\mathbf{k})\mathcal{G}_\ell(i\epsilon,\mathbf{k})\,J^{\omega}_{\mu,\ell m}(\epsilon,\mathbf{k})\,J^{\omega}_{\nu,m\ell}(\epsilon,\mathbf{k})\\ &\quad+\frac{e^2}{2V}\sum_{\mathbf{k}} T\sum_{\epsilon}\epsilon_{\mu\nu}\sum_{\ell,m\neq n}\mathcal{G}_\ell(i\epsilon,\mathbf{k})\mathcal{G}_m(i\epsilon,\mathbf{k})\mathcal{G}_n(i\epsilon,\mathbf{k})J^{\omega}_{\mu,\ell m}(\epsilon,\mathbf{k})\,F^{\epsilon}_{mn}(\epsilon,\mathbf{k})J^{\omega}_{\nu,n\ell}(\epsilon,\mathbf{k})\\ &=\sigma^{1\mathrm{H}}_{12}+\sigma^{2\mathrm{H}}_{12},\end{aligned}\tag{C.11}$$

where we define

$$\begin{aligned}F^{\epsilon}_{\ell m}(\epsilon,\mathbf{k}) &= -\langle u_\ell(\epsilon,\mathbf{k})\mid\partial_\epsilon\hat{\mathcal{G}}^{-1}(\epsilon,\mathbf{k})\mid u_m(\epsilon,\mathbf{k})\rangle\\ &=\langle u_\ell(\epsilon,\mathbf{k})\mid\partial_\epsilon\hat{H}_*(\epsilon,\mathbf{k})\mid u_m(\epsilon,\mathbf{k})\rangle\,,\qquad \ell\neq m\,,\\ J^{\omega}_{\mu,\ell m}(\epsilon,\mathbf{k}) &=\langle u_\ell(\epsilon,\mathbf{k})\mid\hat{J}^{\omega}_{\mu}(\epsilon,\mathbf{k})\mid u_m(\epsilon,\mathbf{k})\rangle\,.\end{aligned}$$

We note that, if we change the Hamiltonian $\hat{H}_*(\epsilon,\mathbf{k})$ into

$$\hat{H}_*(\epsilon,\mathbf{k};\boldsymbol{\kappa})=\hat{H}_*(\epsilon,\mathbf{k})+\boldsymbol{\kappa}\cdot\delta\hat{\boldsymbol{J}}^{\omega}(\epsilon,\mathbf{k})\,,\tag{C.12}$$

then, for $\ell\neq m$ and assuming the parallel-transport gauge,

$$\begin{aligned}F^{\epsilon}_{\ell m}(\epsilon,\mathbf{k}) &= \big(\epsilon_\ell(\epsilon,\mathbf{k})-\epsilon_m(\epsilon,\mathbf{k})\big)\,\langle\partial_\epsilon u_\ell(\epsilon,\mathbf{k})\mid u_m(\epsilon,\mathbf{k})\rangle\\ &=\big(\epsilon_m(\epsilon,\mathbf{k})-\epsilon_\ell(\epsilon,\mathbf{k})\big)\,\langle u_\ell(\epsilon,\mathbf{k})\mid\partial_\epsilon u_m(\epsilon,\mathbf{k})\rangle\,,\\ J^{\omega}_{\mu,\ell m}(\epsilon,\mathbf{k}) &= \big(\epsilon_\ell(\epsilon,\mathbf{k})-\epsilon_m(\epsilon,\mathbf{k})\big)\,\langle\partial_\mu u_\ell(\epsilon,\mathbf{k})\mid u_m(\epsilon,\mathbf{k})\rangle\\ &=\big(\epsilon_m(\epsilon,\mathbf{k})-\epsilon_\ell(\epsilon,\mathbf{k})\big)\,\langle u_\ell(\epsilon,\mathbf{k})\mid\partial_\mu u_m(\epsilon,\mathbf{k})\rangle\,,\end{aligned}$$

with $\partial_\mu=\partial_{k_\mu}+\partial_{\kappa_\mu}$, and, not to weigh too much the notations, we do not explicitly indicate the $\boldsymbol{\kappa}$-dependence since in the final formulas the derivatives are evaluated at $\boldsymbol{\kappa}=\mathbf{0}$.

Coming back to the Hall conductivity, one easily realises that the term in (C.11) where both current vertices correspond to the corrections $\delta\hat{J}^{\omega}_{\mu}(\epsilon,\mathbf{k})$ and $\delta\hat{J}^{\omega}_{\nu}(\epsilon,\mathbf{k})$, see (C.8), vanishes identically, in agreement with the results in Sec. 3. We proceed without making use of this observation until that will become necessary.

Under $\mu\longleftrightarrow\nu$ and $\ell\longleftrightarrow m$, we find that

$$\begin{aligned}\sigma^{1\mathrm{H}}_{12} &= -\frac{e^2}{2V}\sum_{\mathbf{k}} T\sum_{\epsilon}\epsilon_{\mu\nu}\sum_{\ell\neq m}\partial_\epsilon\,\mathcal{G}_\ell(i\epsilon,\mathbf{k})\mathcal{G}_m(i\epsilon,\mathbf{k})\,J^{\omega}_{\mu,\ell m}(\epsilon,\mathbf{k})\,J^{\omega}_{\nu,m\ell}(\epsilon,\mathbf{k})\\ &= -\frac{e^2}{4V}\sum_{\mathbf{k}} T\sum_{\epsilon}\epsilon_{\mu\nu}\sum_{\ell\neq m}\partial_\epsilon\,S_{\ell m}(\epsilon,\mathbf{k})\,\langle\partial_\mu u_\ell(\epsilon,\mathbf{k})\mid u_m(\epsilon,\mathbf{k})\rangle\langle u_m(\epsilon,\mathbf{k})\mid\partial_\nu u_\ell(\epsilon,\mathbf{k})\rangle\,,\end{aligned}\tag{C.13}$$

where [10]

$$\begin{aligned} S_{\ell m}(\epsilon,\mathbf{k}) &= 2\ln\frac{i\epsilon-\epsilon_\ell(\epsilon,\mathbf{k})}{i\epsilon-\epsilon_m(\epsilon,\mathbf{k})} - \frac{i\epsilon-\epsilon_\ell(\epsilon,\mathbf{k})}{i\epsilon-\epsilon_m(\epsilon,\mathbf{k})} + \frac{i\epsilon-\epsilon_m(\epsilon,\mathbf{k})}{i\epsilon-\epsilon_\ell(\epsilon,\mathbf{k})} \\ &= 2i\arg\ln\frac{i\epsilon-\epsilon_\ell(\epsilon,\mathbf{k})}{i\epsilon-\epsilon_m(\epsilon,\mathbf{k})} + 2\ln\left|\frac{i\epsilon-\epsilon_\ell(\epsilon,\mathbf{k})}{i\epsilon-\epsilon_m(\epsilon,\mathbf{k})}\right| + K_{\ell m}(\epsilon,\mathbf{k})\,. \end{aligned} \tag{C.14}$$

Only the imaginary part of $S_{\ell m}(\epsilon,\mathbf{k})$, which is odd in ϵ, contributes to $\sigma^{1\text{H}}_{12}(\mathbf{k})$ in (C.13). Moreover, the argument of the logarithm is discontinuous crossing $\epsilon = 0$ if $\epsilon_\ell(0,\mathbf{k})\,\epsilon_m(0,\mathbf{k}) = \epsilon_\ell(\mathbf{k})\,\epsilon_m(\mathbf{k}) < 0$, specifically,

$$\arg\ln\frac{i\epsilon-\epsilon_\ell(\epsilon,\mathbf{k})}{i\epsilon-\epsilon_m(\epsilon,\mathbf{k})} \xrightarrow[\epsilon\to 0]{} -\pi\,\text{sign}(\epsilon)\Big[\theta\big(-\epsilon_\ell(\mathbf{k})\big)-\theta\big(-\epsilon_m(\mathbf{k})\big)\Big]. \tag{C.15}$$

We hence write

$$\partial_\epsilon S_{\ell m}(\epsilon,\mathbf{k}) \simeq 4\pi\, i\,\delta(\epsilon)\Big[\theta\big(-\epsilon_\ell(\mathbf{k})\big)-\theta\big(-\epsilon_m(\mathbf{k})\big)\Big] + \partial_\epsilon K_{\ell m}(\epsilon,\mathbf{k})\,, \tag{C.16}$$

dropping terms that are either real or odd in ϵ. In the limit of vanishing temperature, we find that, see (C.6),

$$\begin{aligned} \sigma^{1\text{H}}_{12} &= i\,\frac{e^2}{2V}\sum_{\mathbf{k}}\sum_{\ell\neq m}\epsilon_{\mu\nu}\,\frac{\theta\big(-\epsilon_\ell(\mathbf{k})\big)-\theta\big(-\epsilon_m(\mathbf{k})\big)}{\big(\epsilon_\ell(\epsilon,\mathbf{k})-\epsilon_m(\epsilon,\mathbf{k})\big)^2}\,J^\omega_{\mu,\ell m}(\mathbf{k})\,J^\omega_{\nu,m\ell}(\mathbf{k}) \\ &\quad -\frac{e^2}{4V}\sum_{\mathbf{k}}\int\frac{d\epsilon}{2\pi}\,\epsilon_{\mu\nu}\sum_{\ell\neq m}\partial_\epsilon K_{\ell m}(\epsilon,\mathbf{k})\,\langle\partial_\mu u_\ell(\epsilon,\mathbf{k})\mid u_m(\epsilon,\mathbf{k})\rangle\langle u_m(\epsilon,\mathbf{k})\mid\partial_\nu u_\ell(\epsilon,\mathbf{k})\rangle \\ &= \sigma^{\text{H}}_{0\,12} + \sigma^{\epsilon\text{H}}_{12}\,, \end{aligned} \tag{C.17}$$

where $\sigma^{\text{H}}_{0\,12}$ coincides with (8). The contribution $\sigma^{2\text{H}}_{12}$ in (C.11) can be instead written as

$$\begin{aligned} \sigma^{2\text{H}}_{12} &= \frac{e^2}{2V}\sum_{\mathbf{k}}\int\frac{d\epsilon}{2\pi}\,\epsilon_{\mu\nu}\sum_{\ell m n}{}' \mathcal{G}_\ell(i\epsilon,\mathbf{k})\,\mathcal{G}_m(i\epsilon,\mathbf{k})\,\mathcal{G}_n(i\epsilon,\mathbf{k})\,J^\omega_{\mu,\ell m}(\epsilon,\mathbf{k})\,F^\epsilon_{mn}(\epsilon,\mathbf{k})\,J^\omega_{\nu,n\ell}(\epsilon,\mathbf{k}) \\ &\quad -\frac{e^2}{4V}\sum_{\mathbf{k}}\int\frac{d\epsilon}{2\pi}\,\epsilon_{\mu\nu}\sum_{\ell\neq m}\partial_\mu K_{\ell m}(\epsilon,\mathbf{k})\,\langle\partial_\epsilon u_\ell(\epsilon,\mathbf{k})\mid u_m(\epsilon,\mathbf{k})\rangle\langle u_m(\epsilon,\mathbf{k})\mid\partial_\nu u_\ell(\epsilon,\mathbf{k})\rangle \\ &\quad -\frac{e^2}{4V}\sum_{\mathbf{k}}\int\frac{d\epsilon}{2\pi}\,\epsilon_{\mu\nu}\sum_{\ell\neq m}\partial_\nu K_{\ell m}(\epsilon,\mathbf{k})\,\langle\partial_\mu u_\ell(\epsilon,\mathbf{k})\mid u_m(\epsilon,\mathbf{k})\rangle\langle u_m(\epsilon,\mathbf{k})\mid\partial_\epsilon u_\ell(\epsilon,\mathbf{k})\rangle \\ &= \sigma^{3\text{H}}_{12} + \sigma^{\mu\text{H}}_{12} + \sigma^{\nu\text{H}}_{12}\,, \end{aligned} \tag{C.18}$$

where the apex on the sum means that ℓ, m and n are all different, and we replaced $S_{\ell m}(\epsilon,\mathbf{k})$ with $K_{\ell m}(\epsilon,\mathbf{k})$ since the singular term in ϵ does not contribute.

One can readily show, see [10] for details, that, since

$$\mathcal{G}_\ell(i\epsilon,\mathbf{k})\,\mathcal{G}_m(i\epsilon,\mathbf{k})\,\mathcal{G}_n(i\epsilon,\mathbf{k}) = \frac{K_{\ell m}(\epsilon,\mathbf{k})+K_{mn}(\epsilon,\mathbf{k})+K_{n\ell}(\epsilon,\mathbf{k})}{\big(\epsilon_\ell(\epsilon,\mathbf{k})-\epsilon_m(\epsilon,\mathbf{k})\big)\big(\epsilon_m(\epsilon,\mathbf{k})-\epsilon_n(\epsilon,\mathbf{k})\big)\big(\epsilon_n(\epsilon,\mathbf{k})-\epsilon_\ell(\epsilon,\mathbf{k})\big)}\,,$$

then, recalling that $K_{\ell\ell}(\epsilon,\mathbf{k}) = 0$,

$$\begin{aligned} \sigma^{3\text{H}}_{12} = -\frac{e^2}{4V}\sum_{\mathbf{k}}\int\frac{d\epsilon}{2\pi}\,\epsilon_{\mu\nu}\sum_{\ell m}K_{\ell m}(\epsilon,\mathbf{k})\Big\{&\partial_\mu\big(\langle\partial_\epsilon u_\ell(\epsilon,\mathbf{k})\mid u_m(\epsilon,\mathbf{k})\rangle\langle u_m(\epsilon,\mathbf{k})\mid\partial_\nu u_\ell(\epsilon,\mathbf{k})\rangle\big) \\ &+\partial_\epsilon\big(\langle\partial_\nu u_\ell(\epsilon,\mathbf{k})\mid u_m(\epsilon,\mathbf{k})\rangle\langle u_m(\epsilon,\mathbf{k})\mid\partial_\mu u_\ell(\epsilon,\mathbf{k})\rangle\big) \\ &+\partial_\nu\big(\langle\partial_\mu u_\ell(\epsilon,\mathbf{k})\mid u_m(\epsilon,\mathbf{k})\rangle\langle u_m(\epsilon,\mathbf{k})\mid\partial_\epsilon u_\ell(\epsilon,\mathbf{k})\rangle\big)\Big\}\,. \end{aligned} \tag{C.19}$$

It follows that

$$\begin{aligned}
&\sigma^{3\mathrm{H}}_{12}+\sigma^{\mu\mathrm{H}}_{12}+\sigma^{\epsilon\mathrm{H}}_{12}+\sigma^{\nu\mathrm{H}}_{12} \qquad\qquad (C.20)\\
&=-\frac{e^2}{4V}\sum_{\mathbf{k}}\int\frac{d\epsilon}{2\pi}\,\epsilon_{\mu\nu}\sum_{\ell m}\Big\{\partial_\mu\Big(K_{\ell m}(\epsilon,\mathbf{k})\,\langle\partial_\epsilon u_\ell(\epsilon,\mathbf{k})\mid u_m(\epsilon,\mathbf{k})\rangle\langle u_m(\epsilon,\mathbf{k})\mid\partial_\nu u_\ell(\epsilon,\mathbf{k})\rangle\Big)\\
&\qquad+\partial_\epsilon\Big(K_{\ell m}(\epsilon,\mathbf{k})\,\langle\partial_\nu u_\ell(\epsilon,\mathbf{k})\mid u_m(\epsilon,\mathbf{k})\rangle\langle u_m(\epsilon,\mathbf{k})\mid\partial_\mu u_\ell(\epsilon,\mathbf{k})\rangle\Big)\\
&\qquad+\partial_\nu\Big(K_{\ell m}(\epsilon,\mathbf{k})\,\langle\partial_\mu u_\ell(\epsilon,\mathbf{k})\mid u_m(\epsilon,\mathbf{k})\rangle\langle u_m(\epsilon,\mathbf{k})\mid\partial_\epsilon u_\ell(\epsilon,\mathbf{k})\rangle\Big)\Big\},
\end{aligned}$$

is the sum of full derivatives of continuous functions, periodic both in $\mathbf{k}$ and ϵ, noticing that at $\epsilon\to\pm\infty$ we recover the Hartree-Fock results. If both current vertices correspond to the unrenormalized ones, thus $\partial_\mu=\partial_{k_\mu}$ and $\partial_\nu=\partial_{k_\nu}$, (C.20) vanishes identically, consistently with the results of [10]. Therefore, the correction $\delta\sigma^{\mathrm{H}}_{12}$ to $\sigma^{\mathrm{H}}_{0\,12}$ in (C.17) comes from the terms where $\partial_\mu=\partial_{\kappa_\mu}$, and thus $\partial_\nu=\partial_{k_\nu}$, or vice versa, which, being equal, imply, going back to (C.17) and (C.18) and through (C.8), that

$$\begin{aligned}
\delta\sigma^{\mathrm{H}}_{12}&=\frac{e^2}{V}\sum_{\mathbf{k}}\int\frac{d\epsilon}{2\pi}\,\epsilon_{\mu\nu}\sum_{\ell mn}{}'\,\mathcal{G}_\ell(i\epsilon,\mathbf{k})\mathcal{G}_m(i\epsilon,\mathbf{k})\mathcal{G}_n(i\epsilon,\mathbf{k})\,\langle u_\ell(\epsilon,\mathbf{k})\mid\delta\hat{J}^{\omega}_\mu(\epsilon,\mathbf{k})\mid u_m(\epsilon,\mathbf{k})\rangle\\
&\qquad\times\langle u_m(\epsilon,\mathbf{k})\mid\partial_\epsilon\hat{H}_*(\epsilon,\mathbf{k})\mid u_n(\epsilon,\mathbf{k})\rangle\,\langle u_n(\epsilon,\mathbf{k})\mid\partial_{k_\nu}\hat{H}_*(\epsilon,\mathbf{k})\mid u_\ell(\epsilon,\mathbf{k})\rangle\\
&-\frac{e^2}{2V}\sum_{\mathbf{k}}\int\frac{d\epsilon}{2\pi}\,\epsilon_{\mu\nu}\sum_{\ell\neq m}\partial_\epsilon K_{\ell m}(\epsilon,\mathbf{k})\,\langle\delta_\mu u_\ell(\epsilon,\mathbf{k})\mid u_m(\epsilon,\mathbf{k})\rangle\langle u_m(\epsilon,\mathbf{k})\mid\partial_\nu u_\ell(\epsilon,\mathbf{k})\rangle\\
&-\frac{e^2}{2V}\sum_{\mathbf{k}}\int\frac{d\epsilon}{2\pi}\,\epsilon_{\mu\nu}\sum_{\ell\neq m}\delta_\mu K_{\ell m}(\epsilon,\mathbf{k})\,\langle\partial_\epsilon u_\ell(\epsilon,\mathbf{k})\mid u_m(\epsilon,\mathbf{k})\rangle\langle u_m(\epsilon,\mathbf{k})\mid\partial_{k_\nu}u_\ell(\epsilon,\mathbf{k})\rangle\\
&-\frac{e^2}{2V}\sum_{\mathbf{k}}\int\frac{d\epsilon}{2\pi}\,\epsilon_{\mu\nu}\sum_{\ell\neq m}\partial_{k_\nu}K_{\ell m}(\epsilon,\mathbf{k})\,\langle\delta_\mu u_\ell(\epsilon,\mathbf{k})\mid u_m(\epsilon,\mathbf{k})\rangle\langle u_m(\epsilon,\mathbf{k})\mid\partial_\epsilon u_\ell(\epsilon,\mathbf{k})\rangle\,, \qquad (C.21)
\end{aligned}$$

where δ_μ represents the first order expansion in $\delta\hat{J}^{\omega}_\mu(\epsilon,\mathbf{k})$ of eigenstates and eigenvalues of the Hamiltonian (C.12), thus

$$\begin{aligned}
\epsilon_\ell(\epsilon,\mathbf{k})\quad&\to\quad\epsilon_\ell(\epsilon,\mathbf{k})+\delta_\mu\epsilon_\ell(\epsilon,\mathbf{k})=\epsilon_\ell(\epsilon,\mathbf{k})+\delta\hat{J}^{\omega}_{\mu,\ell\ell}(\epsilon,\mathbf{k})\,,\\
\mid u_\ell(\epsilon,\mathbf{k})\rangle\quad&\to\quad\mid u_\ell(\epsilon,\mathbf{k})\rangle+\mid\delta_\mu u_\ell(\epsilon,\mathbf{k})\rangle=\mid u_\ell(\epsilon,\mathbf{k})\rangle+\sum_{m\neq\ell}\frac{\delta\hat{J}^{\omega}_{\mu,m\ell}(\epsilon,\mathbf{k})}{\epsilon_\ell(\epsilon,\mathbf{k})-\epsilon_m(\epsilon,\mathbf{k})}\mid u_m(\epsilon,\mathbf{k})\rangle\,,
\end{aligned}$$

and, we recall,

$$K_{\ell m}(\epsilon,\mathbf{k})=\Big(\epsilon_\ell(\epsilon,\mathbf{k})-\epsilon_m(\epsilon,\mathbf{k})\Big)\Big(\mathcal{G}_\ell(i\epsilon,\mathbf{k})+\mathcal{G}_m(i\epsilon,\mathbf{k})\Big),$$

depends on the eigenvalues. We first note that the last two terms on the right hand side of (C.21) involve the diagonal, intraband, matrix elements of the currents $\delta\hat{J}^{\omega}_\mu(\epsilon,\mathbf{k})$ and $\hat{J}^{q}_\nu(\epsilon,\mathbf{k})$, which cannot contribute to the anomalous Hall conductivity. Therefore, we can safely discard those two terms.

Concerning instead the first two terms on the right hand side of (C.21), we observe that they require, to be finite, phase coherence between the matrix elements connecting two or three distinct bands through different operators. Specifically, one of the matrix elements is that of $\delta\hat{J}^{\omega}_\mu(\epsilon,\mathbf{k})$ in (C.8), which represents an interaction scattering amplitude between a quasiparticle-quasihole pair in bands ℓ and m at finite frequency ϵ, which has to be integrated, and an intraband pair at $\epsilon=0$ on the quasiparticle Fermi surface. It is conceivable, given the decoherence effects brought about by interaction, that such process, just because of the different frequencies of the two pairs, cannot maintain phase coherence upon integrating over frequencies and summing over all bands and momenta. On the contrary, phase coherence is maintained in $\sigma^{\rm H}_{0\,12}$ of (C.17), where both pairs are at $\epsilon = 0$. We shall therefore assume that the first two terms on the right hand side of (C.21) averages at zero, thus leading to $\delta\sigma^{\rm H}_{12} = 0$ and to the desired result

$$\sigma^{\rm H}_{12} = \sigma^{\rm H}_{0\,12} = i\,\frac{e^2}{2V}\sum_{\mathbf{k}}\sum_{\ell\neq m}\epsilon_{\mu\nu}\,\frac{\theta\big(-\epsilon_\ell(\mathbf{k})\big)-\theta\big(-\epsilon_m(\mathbf{k})\big)}{\big(\epsilon_\ell(\epsilon,\mathbf{k})-\epsilon_m(\epsilon,\mathbf{k})\big)^2}\,J^{\omega}_{\mu,\ell m}(\mathbf{k})J^{\omega}_{\nu,m\ell}(\mathbf{k})\,. \tag{C.22}$$

Let us finally discuss the multiband extension of same results in Sec. 3 for the case $\epsilon = 0$, where we believe phase coherence does hold. From the discussion in that section, considering two different states $\alpha \neq \beta$ in the original basis and the Pauli matrices σ_1 and σ_2 in this subspace, thus the off-diagonal generalised Pauli matrices, if

$$\begin{aligned} H_{*,\alpha\beta}(0,\mathbf{k}) &= \left|H_{*,\alpha\beta}(0,\mathbf{k})\right|\,\mathrm{e}^{-i\phi_{\alpha\beta}(\mathbf{k})} \\ &= \left|H_{*,\alpha\beta}(0,\mathbf{k})\right|\big(\cos\phi_{\alpha\beta}(\mathbf{k})\,\sigma_{1,\alpha\beta}+\sin\phi_{\alpha\beta}(\mathbf{k})\,\sigma_{2,\alpha\beta}\big)\,, \end{aligned} \tag{C.23}$$

then we can write

$$\delta J^{\omega}_{\mu,\alpha\beta}(0,\mathbf{k}) = \mathrm{Re}\,\delta J^{\omega}_{\mu,\alpha\beta}(0,\mathbf{k})\big(\cos\phi_{\alpha\beta}(\mathbf{k})\,\sigma_{1,\alpha\beta}+\sin\phi_{\alpha\beta}(\mathbf{k})\,\sigma_{2,\alpha\beta}\big)\,.$$

Therefore, while

$$\begin{aligned} \partial_{k_\mu}H_{*,\alpha\beta}(0,\mathbf{k}) = J^{q}_{\mu,\alpha\beta}(0,\mathbf{k}) & \\ = \partial_{k_\mu}&\left|H_{*,\alpha\beta}(0,\mathbf{k})\right|\big(\cos\phi_{\alpha\beta}(\mathbf{k})\,\sigma_{1,\alpha\beta}+\sin\phi_{\alpha\beta}(\mathbf{k})\,\sigma_{2,\alpha\beta}\big) \\ &-\partial_{k_\mu}\phi_{\alpha\beta}(\mathbf{k})\,\left|H_{*,\alpha\beta}(0,\mathbf{k})\right|\big(\sin\phi_{\alpha\beta}(\mathbf{k})\,\sigma_{1,\alpha\beta}-\cos\phi_{\alpha\beta}(\mathbf{k})\,\sigma_{2,\alpha\beta}\big)\,, \end{aligned}$$

where the interference between the two terms is responsible for the possibly non-trivial topology, $\delta J^{\omega}_{\mu,\alpha\beta}(0,\mathbf{k})$ is instead proportional to only one component of $J^{q}_{\mu,\alpha\beta}(0,\mathbf{k})$, specifically,

$$\delta J^{\omega}_{\mu,\alpha\beta}(0,\mathbf{k}) \propto \partial_{k_\mu}\left|H_{*,\alpha\beta}(0,\mathbf{k})\right|\big(\cos\phi_{\alpha\beta}(\mathbf{k})\,\sigma_{1,\alpha\beta}+\sin\phi_{\alpha\beta}(\mathbf{k})\,\sigma_{2,\alpha\beta}\big)\,,$$

and thus cannot contribute on its own to the anomalous Hall conductivity, which is a further justification of neglecting the second order term in $\delta\hat{\boldsymbol{J}}^{\omega}(\epsilon,\mathbf{k})$.

These properties survive in the diagonal basis, too. It follows that, if we write, for $\ell \neq m$ and using similar conventions as in Sec. 3 but in terms of generalized Pauli matrices,

$$\begin{aligned} J^{q}_{\mu,\ell m}(0,\mathbf{k}) &= J^{q}_{2\mu,\ell m}(0,\mathbf{k})\big(\cos\phi_{\ell m}(\mathbf{k})\,\sigma_{1,\ell m}+\sin\phi_{\ell m}(\mathbf{k})\,\sigma_{2,\ell m}\big) \\ &\quad + J^{q}_{1\mu,\ell m}(0,\mathbf{k})\big(\sin\phi_{\ell m}(\mathbf{k})\,\sigma_{1,\ell m}-\cos\phi_{\ell m}(\mathbf{k})\,\sigma_{2,\ell m}\big) \\ &= \mathrm{J}^{q}_{2\mu,\ell m}(\mathbf{k})+\mathrm{J}^{q}_{1\mu,\ell m}(\mathbf{k})\,, \end{aligned} \tag{C.24}$$

with real $J^{q}_{1\mu,\ell m}(0,\mathbf{k})$ and $J^{q}_{2\mu,\ell m}(0,\mathbf{k})$, in which only the interference between the two may cause a non-trivial topology, then we are allowed to write $\delta J^{\omega}_{\mu,\ell m}(0,\mathbf{k})$ proportional to only one of them, e.g., and without loss of generality,

$$\delta J^{\omega}_{\mu,\ell m}(0,\mathbf{k}) = F^{1}_{2\mu,\ell m}(\mathbf{k})\,\mathrm{J}^{q}_{2\mu,\ell m}(\mathbf{k})\,, \tag{C.25}$$

also with real $F^{1}_{2\mu,\ell m}(\mathbf{k})$. Moreover, if the Fermi surface is isotropic, as we here assume, $F^{1}_{2\mu,\ell m}(\mathbf{k}) = F^{1}_{2,\ell m}(\mathbf{k})$ is independent of the direction μ.

In conclusion, we have shown that the anomalous Hall conductivity $\sigma^{\rm H}_{12}$ in (C.22) reduces to the desired Fermi-liquid expression (8). In other words, we have demonstrated that also in a multi-band Fermi liquid, besides the already well-known single-band case [18, 19], there is perfect coincidence between the physical transport coefficients and those obtained through the interacting Hamiltonian (1) treated within Hartree-Fock plus RPA. Such correspondence holds on condition that the Hartree-Fock Hamiltonian (2) coincides with $\hat{H}_*(\epsilon, \mathbf{k})$ in (B.2) calculated at $\epsilon = 0$, and that the RPA interaction vertices in the dynamic and static limits are equal to the quasiparticle ones, $\hat{\Gamma}^{\omega}_*(0,\mathbf{k};0,\mathbf{k}')$ and $\hat{\Gamma}^{q}_*(0,\mathbf{k};0,\mathbf{k}')$, respectively. The additional assumption that we make besides the standard Fermi liquid ones is that a scattering process between distinct bands induced by the dynamic correction $\delta\hat{\boldsymbol{J}}^{\omega}(\epsilon,\mathbf{k})$ to the current does not maintain phase coherence once summed over frequency, momentum and band indices, thus averaging at zero. The only exceptions are quantities that catch the singularities at zero frequency which arise from the discontinuity of the Green's function phase at $\epsilon = 0$, see, e.g., (C.16), since $\delta\hat{\boldsymbol{J}}^{\omega}(0,\mathbf{k})$ does describe a phase coherent process.

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