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# Jordan meets Freudenthal. A black hole exceptional story

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## Abstract

**Within the extremal black hole attractors arising in ungauged $\mathcal{N} \geqslant 2$-extended Maxwell Einstein supergravity theories in $3+1$ space-time dimensions, we provide an overview of the stratification of the electric-magnetic charge representation space into "large" orbits and related "moduli spaces", under the action of the (continuous limit of the) non-compact *U*-duality Lie group. While each "large" orbit of the *U*-duality supports a class of attractors, the corresponding "moduli space" is the proper subspace of the scalar manifold spanned by those scalar fields on which the Attractor Mechanism is inactive. We present the case study concerning $\mathcal{N} = 2$ supergravity theories with symmetric vector multiplets' scalar manifold, which in all cases (with the exception of the minimally coupled models) have the electric-magnetic charge representation of *U*-duality fitting into a reduced Freudenthal triple system over a cubic (simple or semi-simple) Jordan algebra.**



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## Contents

**1 Introduction** **2**

**2 $\mathcal{N} = 2$ *symmetric* supergravities** **5**

2.1 "Large" *U*-duality orbits 7
2.2 "Moduli spaces" of attractors 8
2.3 "Moduli spaces" of the ADM mass 12

**References** **12**

# 1 Introduction

Within the theory of dynamical (dissipative) systems, an *attractor* is defined by a *fixed point* of the evolution flow of the system itself, describing the equilibrium state and its stability features. In general, when approaching an attractor, the orbits of the dynamical evolution lose all memory of their initial conditions, but nonetheless the overall dynamics remains strictly deterministic.

Within Maxwell-Einstein-scalar theories endowed with local supersymmetry in 3+1 space-time dimensions, attractors were firstly discovered within the class of extremal black hole solutions preserving half of the supersymmetries, in presence of $\mathcal{N} = 2$ spinor supercharges. This led to the discovery of the so-called *Attractor Mechanism* (AM), governing the dynamics of evolution of the scalar fields in the black hole background [1]- [4]. We will now review the basics of AM in such a framework.

As far as propagating (i.e., dynamical) massless fields are concerned, linearly realized $\mathcal{N} = 2$ local supersymmetry in $3 + 1$ space-time dimensions admits three multiplet representations (see *e.g.* [6] for a general treatment and a list of Refs.):

1. one *gravity* multiplet, whose maximal helicity is 2, given by

$$\left(V_\mu^a, \psi^A, \psi_A, A^0\right), \tag{1}$$

   where the *Vielbein* one-form $V^a$ (together with the spin-connection one-form $\omega^{ab}$) relates to the graviton ($a = 0, 1, 2, 3$), $\psi^A, \psi_A$ are $SU(2)$-doublets of spinor one-forms (usually named gravitinos; $A = 1, 2$, with the upper and lower indices respectively denoting right and left chirality, *i.e.* $\gamma_5\psi_A = -\gamma_5\psi^A$), and $A^0$ denotes the Maxwell gauge boson 1-form potential usually named *graviphoton*.

2. $n_V$ *vector* multiplets, whose maximal helicity is 1, given by ($I, i = 1, ..., n_V$)

$$\left(A^I, \lambda^{iA}, \overline{\lambda}^{\bar{i}}_A, z^i\right), \tag{2}$$

   each containing a gauge boson one-form $A^I$ , a $SU(2)$-doublet of zero-form spinors $\lambda^{iA}, \overline{\lambda}^{\bar{i}}_A$ (usually named gauginos), and a complex scalar field (zero-form) $z^i$. The $z^i$'s coordinatize a complex manifold $M_{n_V}$, of complex dimension $n_V$, which is endowed with a projective *special Kähler* structure by supersymmetry.

3. $n_H$ *hypermultiplets*, whose maximal helicity is 1/2, given by ($\alpha = 1, ..., 2n_H$)

$$\left(\zeta_\alpha, \zeta^\alpha, q^u\right), \tag{3}$$

   each containing a pair of zero-form spinors $\zeta_\alpha, \zeta^\alpha$ (named hyperinos), and four real scalar fields $q^u$ ($u = 1, ..., 4n_H$), which coordinatize a quaternionic manifold $\mathcal{Q}_{n_H}$ (of quaternionic dimension $n_H$).

When there is no gauging of any global isometry of $M_{n_V}$ and/or $Q_{n_H}$, the $n_H$ hypermultiplets are not involved in the AM, and they can be completely decoupled from the attractor dynamics in the black hole background. This is a direct consequence of the supersymmetry transformation properties of the zero-form spinor fields: the hyperinos $\zeta_\alpha$'s transformations do not depend on the graviphoton $A^0$ nor on $A^I$'s (i.e., on the Maxwell 1-form potentials), whereas gauginos $\lambda^{iA}$'s ones do depend on the Maxwell potentials. More precisely, when disregarding for simplicity's sake the fermionic and gauging terms, the supersymmetry transformations of hyperinos read [6]

$$\delta\zeta_\alpha = i\mathcal{U}_u^{B\beta}\partial_\mu q^u \gamma^\mu \varepsilon^A \epsilon_{AB}\mathbb{C}_{\alpha\beta}\,, \tag{4}$$

implying that the values of the quaternionic scalar fields $q^u$ in the asymptotical(ly flat,) spacial background are unconstrained, and thus they can vary continuously within $Q_{n_H}$. In other words, the hyperscalars $q^u$'s are *moduli* of the system in absence of gauging.

Consequently, in order to keep the framework as simple as possible, we can totally disregard hypermultiplets, and this actually does not imply any loss of generality, at least when ungauged theories are considered. Thus, we consider asymptotically flat, sphedrically symmetric, static, dyonic extremal black hole solutions of $\mathcal{N}=2$-extended supergravity, in which the gravity multiplet (1) is coupled to $n_V$ vector multiplets (2). Since there is no dependence of the black hole metric on time and azimuthal and polar angles, the unique coordinate characterizing the dynamical evolution of the $n_V$ complex scalar fields (one for each vector multiplet) is the radial coordinate: the AM states that, when approaching the event horizon of the black hole, one can always find a solution of the scalar flow such that the scalars dynamically run into fixed points, acquiring values which only depend on (the ratios of) the electric and magnetic charges of the black hole (respectively denoted by $q_\Lambda$ and $p^\Lambda$, with $\Lambda = 0, 1, ..., n_V$), which are conserved quantities due to the overall $U(1)^{n_V}$ gauge symmetry of the system itself and are arranged into the symplectic vector

$$\mathcal{Q} := (p^\Lambda, q_\Lambda)^T\,. \tag{5}$$

Such near-horizon configurations of the scalar fields are completely independent on the boundary conditions of the corresponding dynamics, namely on the spacial asymptotical values of the scalars. Consequently, the dynamical system describing the scalar flow completely loses memory of its initial data, because the dynamical evolution is "attracted" to some fixed configuration points, depending on the electric and magnetic charges *only*. Note that there are no attractors in "pure" $\mathcal{N}=2$ supergravity, since the $\mathcal{N}=2$ gravity multiplet (1) has no scalar fields (in fact, the Reissner-Nordström extremal black hole background is scalarless).

In presence of (linearly realized) local supersymmetry, extremal black holes can be interpreted as BPS (Bogomol'ny-Prasad-Sommerfeld)-saturated [7] solutions, in the low-energy, effective field theory limit of higher-dimensional, UV-complete theories, such as $(9+1)$-dimensional superstrings or $(10+1)$-dimensional $M$-theory [8]. As class of solutions to the Maxwell-Einstein equations of motion, the extremal black holes under considerations are determined by their (asymptotical) ADM mass [9], by the electrical and magnetic charges (defined by integrating the fluxes of related field strengths' 2-forms over a two-sphere at infinity), and by the asymptotical values of the $n_V$ complex scalar fields. Thus, the AM implies that the extremal black holes become "bald", i.e. they lose all their "scalar hair" in the near-horizon limit; in other words, when the extremal black hole metric approaches the conformally flat Bertotti-Robinson $AdS_2 \otimes S^2$ metric [10,11], it is completely characterized only by electric and magnetic charges, but *not* by the continuously-varying asymptotical values of the scalar fields.

A major breakthrough in the study of AM was achieved in [5], in which the fixed points of the scalar dynamics in the extremal black hole background were characterized as critical points of a suitably defined "black hole effective potential" $V_{BH}$, in general being a strictly positive definite function of the $2n_V$ real scalars $\phi^a$ (corresponding to $n_V$ complex scalar fields) and of the $2n_V$ electric and magnetic (real) charges: $V_{BH} = V_{BH}(\phi, \mathcal{Q})$ For a fixed set of e.m. charges $\mathcal{Q}$ (5), the non-degenerate critical points of $V_{BH}$ in $M_{n_V}$, i.e. those points in $M_{n_V}$ such that

$$\frac{\partial V_{BH}}{\partial \phi^a} = 0: \quad V_{BH}|_{\frac{\partial V_{BH}}{\partial \phi}=0} > 0\,, \quad \forall\, a = 1, ..., 2n_V\,, \tag{6}$$

completely determine the values of the scalar fields in the near-horizon limit, which depend on the electric and magnetic charges of the black hole only. The (semi)classical Bekenstein-Hawking entropy ($S_{BH}$) - area ($A_H$) formula [12]- [15] yields the extremal black hole entropy $S_{BH}$ to be given by ($\pi$ times) the critical value of $V_{BH}$ itself:

$$S_{BH} = \pi \frac{A_H}{4} = \pi \; V_{BH}|_{\frac{\partial V_{BH}}{\partial \phi}=0} \; . \tag{7}$$

These result reduce the study of extremal black hole attractors to the study and classification of the various classes of critical points of $V_{BH}$ which yield a non-vanishing critical value of $V_{BH}$ itself; as we will see below, each of these classes is in $1:1$ correspondence with a $U$-orbit supporting an attractor, and thus to an attractor "moduli space".

The fluxes (over $S^2_\infty$, which exists because of the spherical symmetry of the black hole metric) of the Maxwell 2-form field strengths (and of their Lagrangian duals) determine the electric-magnetic charges $\mathcal{Q}$ (5) of the black hole itself, which are $2(n_V+1)$ conserved quantities, where $n_V$ is the number of vector multiplets. The "+1" corresponds to the contribution of the graviphoton Maxwell field. In the limit of real values (which is customarily taken within supergravity, thus disregarding charge quantization, and in particular the Dirac-Schwinger-Zwanzinger quantizations condition for dyons), the $2(n_V+1)$ e.m. charges coordinatize a vector space which is the representation space $\mathcal{Q} \equiv \mathbf{R}_{\mathcal{Q}}$ of the $U$-duality Lie group $G$. Onto $\mathbf{R}_{\mathcal{Q}}$, $G$ acts as a (maximal, non-symmetric) subgroup of $Sp\,(2(n_V+1),\mathbb{R})$, the split real form of the Lie group whose Lie algebra is $\mathfrak{c}_{n_V+1}$:

$$G \overset{\mathbf{R}_{\mathcal{Q}}}{\subsetneq} Sp\,(2(n_V+1),\mathbb{R}) \; . \tag{8}$$

Equivalently, one can state that the embedding (8), whose relevance in field theory was firstly studied by Gaillard and Zumino [16], is a consequence of the fact that the (not necessarily irreducible) $G$-representation $\mathbf{R}_{\mathcal{Q}}$ is anti-self-conjugated (i.e., symplectic), by applying a general theorem of Dynkin [17]. Moreover, it should be pointed out that what we are naming as $U$-duality Lie group $G \equiv G_{\mathbb{R}}$ is actually the (unquantized,) continuous version of the actual $U$-duality, stringy group $G_{\mathbb{Z}}$ [18]. This is consistent with the aforementioned (semi-)classical limit of real charges, also taken into account by the fact that we consider $Sp\,(2(n_V+1),\mathbb{R})$, and not $Sp\,(2(n_V+1),\mathbb{Z})$.

Since the action of $G$ onto $\mathbf{R}_{\mathcal{Q}}$ is in general *non-transitive*, the linear representation vector space $\mathbf{R}_{\mathcal{Q}}$ gets stratified into disjoint classes of orbits under the action of $G$ itself [19–21]: in general, a $G$-orbit $\mathcal{O}$ is a (usually non-symmetric) homogeneous space of $G$,

$$\mathcal{O} \simeq \frac{G}{\mathcal{H}} \subsetneq \mathbf{R}_{\mathcal{Q}} \, , \tag{9}$$

where the isotropy Lie group $\mathcal{H}$ is a (generally non-maximal nor compact) subgroup of $G$ itself, and it is named *stabilizer* of $\mathcal{O}$.

A remarkable fact, stemming from the classical invariant theory applied to the mathematical structure of Maxwell-Einstein-scalar theories, is the following: in all[1] $\mathcal{N}=2$ supergravity theories with homogeneous *symmetric* (vector multiplets') scalar manifolds in $3+1$ space-time dimensions, the pair $\left(G,\mathbf{R}_{\mathcal{Q}}\right)$ is (a suitable real form of) a $\theta$-group *à la Vinberg* [22], namely the number of nilpotent $G$-orbits in $\mathbf{R}_{\mathcal{Q}}$ is finite, and the ring of $G$-invariant polynomials on $\mathbf{R}_{\mathcal{Q}}$ is *finitely generated* (with no *syzygies*) by a unique primitive, homogeneous polynomial $\mathcal{I} \equiv \mathcal{I}(\mathcal{Q})$, of degree two or four in $\mathcal{Q}$ (which we will denotes as $\mathcal{I}_2$ resp. $\mathcal{I}_4$); see e.g. Table II of [23], and Refs. therein. In all these cases, the formula (7) acquires a manifestly $G$-invariant

[1]The same holds for $\mathcal{N} \geqslant 3$-extended supergravity theories, which however we will not treat here.

form,

$$S_{BH} = \pi \frac{A_H}{4} = \pi \begin{cases} |\mathcal{I}_2(\mathcal{Q})| \,, \\ or \\ \sqrt{|\mathcal{I}_4(\mathcal{Q})|} \,. \end{cases} \tag{10}$$

Interestingly, formula (10) relates the Bekenstein-Hawking entropy of extremal black holes to the theory of the aforementioned distinguished class of $\theta$-groups, which can actually be identified as Lie groups of type $E_7$ *à la Brown* of non-degenerate (when $\mathcal{I} = \mathcal{I}_4$) [24] or degenerate (when $\mathcal{I} = \mathcal{I}_2$) [25, 26] type. After Brown [24], non-degenerate groups of type $E_7$ can always be characterized as automorphism groups of *Freudenthal triple systems* (which in turn can be of reduced or non-reduced type; see below).

Clearly, the value acquired by $\mathcal{I}$ is constant along any $G$-orbit. When $\mathcal{I} \neq 0$, the corresponding (generic, open, non-nilpotent) $G$-orbit supports a "large" extremal black hole, which has $S_{BH} \neq 0$, and thus $A_H \neq 0$, at the two-derivative (Einstein) level; on the other hand, when $\mathcal{I} = 0$, the corresponding (nilpotent) $G$-orbit supports a "small" extremal black hole, which has $S_{BH} = 0$, and thus $A_H = 0$, at the two-derivative (Einstein) level: thus, such a "small" black hole is intrinsically quantum, since it needs of an higher-derivative theory of gravity (such as the ones occurring in string effective actions) for a sensible description as solution within a Lagrangian theory.

Moreover, in all the above cases, a manifestly $G$-invariant presentation of the $G$-orbit stratification of $\mathbf{R}_Q$ is given by the $1:1$ correpondence between $G$-invariant sets of algebro-differential constraints on $\mathcal{I}(\mathcal{Q})$ and the various (classes of isomorphic) $G$-orbits $\mathcal{O}$'s. Over $\mathbb{C}$, all "large" $G$-orbits, which are level hypersurfaces in $\mathbf{R}_Q$, are isomorphic, defining the generic, open orbit; however, over $\mathbb{R}$, different real forms (of Riemannian or pseudo-Riemannian type) exist, distinguished by sign($\mathcal{I}$), but possibly (when $G$ is non-degenerate and non-split) also by further $G$-invariant "finer" constraints on $\mathcal{I}$. On the other hand, when $G$ is non-degenerate, the stratification of "small" (i.e. nilpotent) $G$-orbits over $\mathbb{C}$ may involve $G$-invariant differential constraints on $\mathcal{I}$, and, when $G$ is non-split, finer splittings of the $G$-orbit stratification may occur over $\mathbb{R}$. For instance, when $\mathcal{I} = \mathcal{I}_4$ (i.e., for $G$ being non-degenerate of type $E_7$), the stratification of nilpotent $G$-orbits is given by [27]

$$\begin{array}{lll} \text{nilp. } G\text{-orbit} & G\text{-inv. constraint} & \text{rank}_{FTS}(\mathcal{Q}) \\ \mathcal{O}_3: & \mathcal{I}_4 = 0\,, & 3\,, \\ \mathcal{O}_2: & \partial\mathcal{I}_4 = 0\,, & 2\,, \\ \mathcal{O}_1: & \partial^2\mathcal{I}_4\big|_{\mathbf{Adj}(G)} = 0\,, & 1\,, \end{array} \tag{11}$$

where $\text{rank}_{FTS}(\mathcal{Q})$ indicates the $G$-invariant rank[2] of $\mathcal{Q} \equiv \mathbf{R}_Q$ as element of a (reduced) *Freudenthal triple system* [28, 29], which in turn is constructed over a rank-3 Jordan algebra (which, for $\mathcal{N} = 2$ *symmetric* supergravities, are simple or semi-simple; see table 2). Over $\mathbb{R}$, when $G$ is split, the stratification of nilpotent orbits is still given by (11), whereas when $G$ is minimally non-compact, each of the $\mathcal{O}_3$ and $\mathcal{O}_2$ split into two $G$-orbits. Note that $\mathcal{O}_1$, which is the minimal, highest weight $G$-orbit, has the *largest* stabilizer and it is always *unique*.

## 2 $\mathcal{N} = 2$ *symmetric* supergravities

$\mathcal{N} = 2$-extended Maxwell-Einstein supergravity theories [33]- [35] with homogeneous symmetric special Kähler vector multiplets' scalar manifolds will henceforth be shortly referred to as *symmetric* Maxwell-Einstein supergravities. The Riemannian, non-compact, symmetric

[2] If $\mathcal{Q}$ belongs to a "large" $G$-orbit, i.e. when it is such that $\mathcal{I}_4(\mathcal{Q}) \neq 0$, then $\text{rank}_{FTS}(\mathcal{Q}) = 4$.

Table 1: Riemannian symmetric non-compact special Kähler spaces (*alias* vector multiplets' scalar manifolds of the *symmetric* $\mathcal{N}=2$, $D=4$ Maxwell Einstein supergravity theories). $r$ denotes the geodesic rank of the manifold, whereas $n_V$ stands for the number of vector multiplets.

| | $\frac{G}{H_0 \times U(1)}$ | $r$ | $\dim_{\mathbb{C}} \equiv n_V$ |
|---|---|---|---|
| *minimal coupling* $n \in \mathbb{N}$ | $\overline{\mathbb{CP}}^n \equiv \frac{SU(1,n)}{U(1)\times SU(n)}$ | 1 | $n$ |
| $\mathbb{R} \oplus \mathbf{\Gamma}_{1,n-1}$, $n \in \mathbb{N}$ | $\frac{SU(1,1)}{U(1)} \times \frac{SO(2,n)}{SO(2)\times SO(n)}$ | $2\ (n=1)$ <br> $3\ (n \geqslant 2)$ | $n+1$ |
| $J_3^{\mathbb{O}}$ | $\frac{E_{7(-25)}}{E_{6(-78)}\times U(1)}$ | 3 | 27 |
| $J_3^{\mathbb{H}}$ | $\frac{SO^*(12)}{U(6)}$ | 3 | 15 |
| $J_3^{\mathbb{C}}$ | $\frac{SU(3,3)}{S(U(3)\times U(3))}$ | 3 | 9 |
| $J_3^{\mathbb{R}}$ | $\frac{Sp(6,\mathbb{R})}{U(3)}$ | 3 | 6 |
| $\mathbb{R}$ | $\frac{SL(2,\mathbb{R})}{U(1)}$ | 1 | 1 |

special Kähler manifolds have the general coset structure

$$M_{n_V} := \frac{G}{H_0 \times U(1)}, \tag{12}$$

where $H_0 \times U(1)$ is the maximal compact subgroup (mcs) of the $U$-duality group $G$. They have been classified in [30, 31] (see *e.g.* [32] for a quite recent account), and they are reported in Table 1. All the corresponding supergravity theories actually have a five-dimensional origin, since they can be obtained from "parent" (minimally supersymmetric) $\mathcal{N}=2$ supergravities in $4+1$ space-time dimensions, by compactifying *à la Kaluza-Klein* on $S^1$, and retaining the massless sector. This is reflected in the fact that all such theories are endowed with a holomorphic prepotential function which, after projectivization of the coordinates, is a homogeneous cubic polynomial [33]- [35]. The unique exception is provided by the so-called *Luciani theories* [36], which do not have a five-dimensional origin and correspond to the *minimal coupling* of vector multiplets to $\mathcal{N}=2$ supergravity. The corresponding special Kähler manifolds are all symmetric spaces, all with geodesic rank one, and they are nothing but the Riemannian non-compact versions of the $n_V$-dimensional complex projective spaces $\overline{\mathbb{C}P}^{n_V}$ (see e.g. [37, 38]); in these theories, the prepotential is a homogeneous quadratic polynomial, and thus the trilinear coupling of $\mathcal{N}=2$ supergravity, expressed by the so-called $C$-tensor of special geometry, vanishes: $C_{ijk}=0$.

As unraveled for the first time in [33]- [35], the cubic prepotentials of *symmetric* Maxwell-Einstein supergravities are all related to the degree-3 (cubic) norm defined in the corresponding rank-3 Jordan algebra. The sequence of factorized spaces in the third row of Table 1, which is usually referred to as the *generic Jordan family*, is related to the semi-simple rank-3 Jordan algebras $\mathbb{R} \oplus \mathbf{\Gamma}_{1,n-1}$, where $\mathbf{\Gamma}_{1,n-1}$ stands for the degree-2 Jordan algebra with a quadratic form of Lorentzian signature $(1, n-1)$ (*spin factors*) [39]. The complex dimension of the corresponding special Kähler manifold[3]

$$\frac{SL(2,\mathbb{R})}{U(1)} \times \frac{SO(2,n)}{SO(2) \times SO(n)}, \tag{13}$$

is $n+1$, and its geodesic rank is 1+min$(2,n)$. On the other hand, the four isolated *"magic"* theories are based on the four simple rank-3 Jordan algebras $J_3^{\mathbb{O}}$, $J_3^{\mathbb{H}}$, $J_3^{\mathbb{C}}$ and $J_3^{\mathbb{R}}$, which can be realized as generalized matrix algebras of $3 \times 3$ Hermitian matrices over the four Huwitz's normed division algebras $\mathbb{O}$ (octonions), $\mathbb{H}$ (quaternions), $\mathbb{C}$ (complex numbers) and $\mathbb{R}$ (real numbers) [33–35, 39, 41–43]. The name *"magic"* is due to the fact that the Lie algebras of their $U$-duality groups in $D = 2{+}1, 3{+}1$ and $4{+}1$ space-time dimensions fit into the celebrated *Magic Square* of Freudenthal and Tits [44–46]. By defining $A \equiv \dim_{\mathbb{R}} \mathbb{A}$ $(= 8, 4, 2, 1$ for $\mathbb{A} = \mathbb{O}, \mathbb{H}, \mathbb{C}, \mathbb{R}$, respectively), the complex dimension of the symmetric cosets of the "magic" supergravities is $3(A+1)$. *Last but not least*, the special Kähler scalar manifold of the so-called $T^3$-model is the rank-1 coset $\frac{SL(2,\mathbb{R})}{U(1)}$ based on the cubic prepotential $F = T^3$ and related to the simplest cubic Jordan algebra, given by the real numbers, with the cubic norm simply given by the cube power. This model has a unique vector multiplet coupled to $\mathcal{N} = 2$ supergravity, and it can be obtained by dimensional reduction of five-dimensional minimal "pure" supergravity.

### 2.1 "Large" $U$-duality orbits

The classification of $U$-duality orbits supporting "large" extremal black holes in symmetric Maxwell-Einstein supergravities in $3+1$ space-time dimensions has been carried out in [37], and it is reported in[4] Table 2.

Given the scalar manifold (12), the U-duality orbits which support ($\frac{1}{2}$-)BPS-saturated black holes , i.e. which preserve the maximal ($\frac{1}{2}$) amount of supersymmetry, has structure

$$\mathcal{O}_{BPS} = \frac{G}{H_0}, \text{ with } H_0 \times U(1) \overset{\text{mcs}}{\subsetneq} G. \tag{14}$$

As discovered in [37], there are other two non-isomorphic classes of $U$-duality orbits, both supporting extremal black hole attractors which are non-supersymmetric (i.e., which dor not saturate the BPS bound [7]). The first non-supersymmetric (non-BPS) orbit has non-vanishing $\mathcal{N} = 2$ central charge at the horizon ($Z_H \neq 0$), with coset structure

$$\mathcal{O}_{nBPS, Z_H \neq 0} = \frac{G}{\widehat{H}}, \text{ with } \widehat{H} \times SO(1,1) \subsetneq G, \tag{15}$$

where $\widehat{H}$ denotes the $U$-duality group of the corresponding parent theory in $4+1$ space-time dimensions, and $SO(1,1)$ corresponds to the radius of the circle $S^1$ in the Kaluza-Klein reduction from five to four dimensions. The second class of non-BPS $U$-duality orbits has vanishing central charge at the black hole horizon: $Z_H = 0$, with coset structure

$$\mathcal{O}_{nBPS, Z_H = 0} = \frac{G}{\widetilde{H}}, \text{ with } \widetilde{H} \times U(1) \subsetneq G. \tag{16}$$

[3]This is the unique special Kähler manifold which is the product of two irreducible spaces, as proved in [40].

[4]In the present report, we will not consider the highly-degenerate case given by the so-called $T^3$-model, for which the reader is addressed to [47], and to Refs. therein.

Table 2: Large $G$-orbits of *symmetric* $\mathcal{N}=2$, $D=4$ Maxwell-Einstein supergravities. They all support extremal black hole attractors, with different supersymmetry-preserving features.

| | $\frac{1}{2}$-BPS orbit $\mathcal{O}_{\frac{1}{2}-BPS}=\frac{G}{H_0}$ | nBPS $Z_H\neq 0$ orbit $\mathcal{O}_{nBPS,Z_H\neq 0}=\frac{G}{\widehat{H}}$ | nBPS $Z_H=0$ orbit $\mathcal{O}_{nBPS,Z_H=0}=\frac{G}{\widetilde{H}}$ |
|---|---|---|---|
| *minimal coupling* $n\in\mathbb{N}$ | $\frac{SU(1,n)}{SU(n)}$ | – | $\frac{SU(1,n)}{SU(1,n-1)}$ |
| $\mathbb{R}\oplus\mathbf{\Gamma}_{1,n-1}$ $n\in\mathbb{N}$ | $SU(1,1)\times\frac{SO(2,n)}{SO(2)\times SO(n)}$ | $SU(1,1)\times\frac{SO(2,n)}{SO(1,1)\times SO(1,n-1)}$ | $SU(1,1)\times\frac{SO(2,n)}{SO(2)\times SO(2,n-2)}$ |
| $J_3^{\mathbb{O}}$ | $\frac{E_{7(-25)}}{E_6}$ | $\frac{E_{7(-25)}}{E_{6(-26)}}$ | $\frac{E_{7(-25)}}{E_{6(-14)}}$ |
| $J_3^{\mathbb{H}}$ | $\frac{SO^*(12)}{SU(6)}$ | $\frac{SO^*(12)}{SU^*(6)}$ | $\frac{SO^*(12)}{SU(4,2)}$ |
| $J_3^{\mathbb{C}}$ | $\frac{SU(3,3)}{SU(3)\times SU(3)}$ | $\frac{SU(3,3)}{SL(3,\mathbb{C})}$ | $\frac{SU(3,3)}{SU(2,1)\times SU(1,2)}$ |
| $J_3^{\mathbb{R}}$ | $\frac{Sp(6,\mathbb{R})}{SU(3)}$ | $\frac{Sp(6,\mathbb{R})}{SL(3,\mathbb{R})}$ | $\frac{Sp(6,\mathbb{R})}{SU(2,1)}$ |

Note that $\widehat{H}$ and $\widetilde{H}$ are the only two non-compact forms of $H_0$ embedded (with rank-1 commutant) into $G$ itself. Thus, the group embedding in the r.h.s. of (15) and (16) are both maximal and symmetric (see *e.g.* [48–50]).

While $H_0$ is a real compact Lie group (stabilizing the BPS "large" orbit (14)), the groups $\widehat{H}$ and $\widetilde{H}$, respectively stabilizing the non-BPS "large" orbits with $Z_H\neq 0$ (15) and $Z_H=0$ (16), are non-compact, and thus they will admit a proper maximal compact subgroup, which we denote with $\widehat{h}$ resp. $\widetilde{h}$:

$$\widehat{h}=\mathrm{mcs}\left(\widehat{H}\right), \qquad \widetilde{h}=\mathrm{mcs}\left(\widetilde{H}\right). \tag{17}$$

## 2.2 "Moduli spaces" of attractors

For symmetric $\mathcal{N}=2$ supergravities, general results on the rank $\mathfrak{r}$ of the $2n_V\times 2n_V$ Hessian matrix $\mathbf{H}$ of the effective black hole potential $V_{BH}(\phi,\mathcal{Q})$ at its critical points are known (see e.g. [37] and [51]).

The BPS (non-degenerate) critical points of $V_{BH}$ are stable, and thus the Hessian matrix at BPS critical points $\mathbf{H}_{BPS}$ has no massless modes [5], and its rank is maximal: $\mathfrak{r}_{BPS}=2n_V$. Furthermore, the analysis carried out in [37] showed that for the other two classes of non-BPS critical points of $V_{BH}$, the rank of $\mathbf{H}$ is model-dependent:

$$\overline{\mathbb{CP}}^n : \ \mathfrak{r}_{nBPS,Z_H=0} = 2\,, \tag{18}$$

$$\mathbb{R} \oplus \mathbf{\Gamma}_{1,n-1} : \begin{cases} \mathfrak{r}_{nBPS,Z_H\neq0} = n+2\,, \\ \\ \mathfrak{r}_{nBPS,Z_H=0} = 6\,, \end{cases} \tag{19}$$

$$J_3^{\mathbb{A}} : \begin{cases} \mathfrak{r}_{nBPS,Z_H\neq0} = 3A+4\,, \\ \\ \mathfrak{r}_{nBPS,Z_H=0} = 2A+6\,. \end{cases} \tag{20}$$

Correspondingly, the number $\sharp$ of massless Hessian modes for the various models is given by

$$\sharp := 2n_V - \mathfrak{r}\,, \tag{21}$$

and thus

$$\overline{\mathbb{CP}}^n : \ \sharp_{nBPS,Z_H=0} = 2\left(n_V - 1\right)\,, \tag{22}$$

$$\mathbb{R} \oplus \mathbf{\Gamma}_{1,n-1} : \begin{cases} \sharp_{nBPS,Z_H\neq0} = n\,, \\ \\ \sharp_{nBPS,Z_H=0} = 2n-4\,, \end{cases} \tag{23}$$

$$J_3^{\mathbb{A}} : \begin{cases} \sharp_{nBPS,Z_H\neq0} = 3A+2\,, \\ \\ \sharp_{nBPS,Z_H=0} = 4A\,. \end{cases} \tag{24}$$

From previous statements, it also holds that

$$\sharp_{BPS} = 0\,, \tag{25}$$

for all $\mathcal{N} = 2$ theories, regardless the properties of the special Kähler vector multiplets' scalar manifold.

Let us start by recalling that $V_{BH}$ is defined as

$$V_{BH}\left(\phi, \mathcal{Q}\right) := -\frac{1}{2}\mathcal{Q}^T \mathbf{M}(\phi)\mathcal{Q}\,, \tag{26}$$

where $\phi$ denotes the $2n_V$ real scalar fields parametrising the special Kähler scalar manifold $\frac{G}{H_0\times U(1)}$, and $\mathcal{Q}$ is the symplectic vector of e.m. black hole charges sitting in the $G$-irrepr. $\mathbf{R}_{\mathcal{Q}}$ of the $U$-duality group $G$. Moreover, $\mathbf{M}(\phi)$ is the $2\left(n_V+1\right)\times 2\left(n_V+1\right)$ real, symmetric and symplectic matrix defined as [52–54]

$$\mathbf{M}(\phi) = -\left(\mathbf{L}\mathbf{L}^T\right)^{-1}\,, \tag{27}$$

where $\mathbf{L} = \mathbf{L}(\phi)$ is coset representative of $\frac{G}{H_0\times U(1)}$, *i.e.* a local section of the principal $G$-bundle over the special Hodge-Kähler scalar manifold $\frac{G}{H_0\times U(1)}$, with structure group $H_0 \times U(1)$.

The action of an element $g \in G$ on $V_{BH}$ (26) is such that

$$G : V_{BH}\left(\phi, \mathcal{Q}\right) \mapsto V_{BH}\left(\phi_g, \mathcal{Q}^g\right) = V_{BH}\left(\phi_g, \left(g^{-1}\right)^T \mathcal{Q}\right)\,, \tag{28}$$

thus, $V_{BH}$ is not $G$-invariant, because its coefficients (given by the components of $\mathcal{Q}$) do not in general remain the same. The situation changes if one restricts $g$ to $g_{\mathcal{Q}} \in \mathcal{H}$, i.e. if one restricts to the stabilizer $\mathcal{H}$ of the "large" $G$-orbits $\mathcal{O} \simeq \frac{G}{\mathcal{H}} \subsetneq \mathbf{R}_{\mathcal{Q}}$ (cf. (9)) to which $\mathcal{Q}$ belongs. In such a case, by definition of $\mathcal{H}$:

$$\mathcal{Q}^{g_{\mathcal{Q}}} = \mathcal{Q} \tag{29}$$

$$\Downarrow$$

$$\mathcal{H} : V_{BH}(\phi, \mathcal{Q}) \mapsto V_{BH}\left(\phi_{g_{\mathcal{Q}}}, \mathcal{Q}^{g_{\mathcal{Q}}}\right) = V_{BH}\left(\phi_{g_{\mathcal{Q}}}, \left(g_{\mathcal{Q}}^{-1}\right)^{T} \mathcal{Q}\right) = V_{BH}\left(\phi_{g_{\mathcal{Q}}}, \mathcal{Q}\right) \simeq V_{BH}(\phi, \mathcal{Q})\,. \tag{30}$$

Then, it is natural to split the $2n_V$ real scalar fields $\phi$ as $\phi = \left\{\phi_{\mathcal{Q}}, \breve{\phi}_{\mathcal{Q}}\right\}$, where

- $$\phi_{\mathcal{Q}} \in \frac{\mathcal{H}}{\mathrm{mcs}(\mathcal{H})} \subsetneq M_{n_V}\,, \tag{31}$$

  where we denote
  $$\frac{\mathcal{H}}{\mathrm{mcs}(\mathcal{H})} =: \mathcal{M}_{\mathcal{Q}}\,, \tag{32}$$

- $\breve{\phi}_{\mathcal{Q}}$ coordinatize the complement of $\mathcal{M}_{\mathcal{Q}}$ in $M_{n_V}$:
  $$\breve{\phi}_{\mathcal{Q}} \in M_{n_V} \backslash \mathcal{M}_{\mathcal{Q}}\,. \tag{33}$$

One can then define

$$V_{BH,crit}\left(\phi_{\mathcal{Q}}, \mathcal{Q}\right) := V_{BH}(\phi, \mathcal{Q})|_{\frac{\partial V_{BH}}{\partial \breve{\phi}_{\mathcal{Q}}}=0} (\neq 0)\,, \tag{34}$$

as the values of $V_{BH}$ along the equations of motion for the scalars $\breve{\phi}_{\mathcal{Q}}$. Thus, (30) implies the invariance of $V_{BH,crit}\left(\phi_{\mathcal{Q}}, \mathcal{Q}\right)$ under $\mathcal{H}$:

$$\mathcal{H} : V_{BH,crit}\left(\phi_{\mathcal{Q}}, \mathcal{Q}\right) \mapsto V_{BH,crit}\left(\left(\phi_{\mathcal{Q}}\right)_{g_{\mathcal{Q}}}, \mathcal{Q}\right) \simeq V_{BH,crit}\left(\phi_{\mathcal{Q}}, \mathcal{Q}\right)\,. \tag{35}$$

Finally, it is crucial to observe that $\mathcal{H}$, except for the ($\frac{1}{2}$-)BPS "large" $G$-orbit, is generally a *non-compact* real Lie group. This implies that $V_{BH}$ at its critical points is independent on the subset of scalar fields

$$\phi_{\mathcal{Q}} \in \mathcal{M}_{\mathcal{Q}} \subsetneq M_{n_V}\,, \tag{36}$$

i.e. on those scalar fields belonging to the homogeneous symmetric submanifold $\mathcal{M}_{\mathcal{Q}} \subsetneq M_{n_V}$, which thus be regarded as the "moduli space" of the attractor solutions supported by the charge orbit $\mathcal{O} \simeq \frac{G}{\mathcal{H}} \subsetneq \mathbf{R}_{\mathcal{Q}}$. Thus,

$$\frac{\partial V_{BH,crit}\left(\phi_{\mathcal{Q}}, \mathcal{Q}\right)}{\partial \phi_{\mathcal{Q}}} = 0 \Rightarrow V_{BH}(\phi, \mathcal{Q})|_{\frac{\partial V_{BH}}{\partial \breve{\phi}_{\mathcal{Q}}}=0} =: V_{BH,crit}(\mathcal{Q})\,, \tag{37}$$

or, equivalently:

$$V_{BH}(\phi, \mathcal{Q})|_{\frac{\partial V_{BH}}{\partial \phi}=0} = V_{BH}(\phi, \mathcal{Q})|_{\frac{\partial V_{BH}}{\partial \breve{\phi}_{\mathcal{Q}}}=0} =: V_{BH,crit}(\mathcal{Q})\,. \tag{38}$$

By using this line of reasoning, in [55] (see also [56]) it was proved that, remarkably, the rank of **H** corresponds to all positive eigenvalues (i.e., stable directions in the scalar manifold), and also that the massless modes of **H** are actually "flat" directions of $V_{BH}$ at the corresponding classes of its critical points. Thus, such "flat" directions of the critical values of $V_{BH}$ span some "moduli spaces" of the attractor solutions [55], corresponding to those scalar degrees of

Table 3: "Moduli spaces" of non-BPS $Z_H \neq 0$ extremal black hole attractors in $\mathcal{N}=2$, $D=4$ *symmetric* Maxwell-Einstein supergravities. They are the real special (vector multiplets') scalar manifolds of the corresponding $\mathcal{N}=2$, $D=5$ symmetric "parent" supergravity theory.

| | $\frac{\widehat{H}}{mcs(\widehat{H})}$ | $r$ | $\dim_{\mathbb{R}}$ |
|---|---|---|---|
| $\mathbb{R}\oplus\mathbf{\Gamma}_{1,n-1},\ n\in\mathbb{N}$ | $SO(1,1)\times\frac{SO(1,n-1)}{SO(n-1)}$ | $1\ (n=1)$<br>$2\ (n\geqslant 2)$ | $n$ |
| $J_3^{\mathbb{O}}$ | $\frac{E_{6(-26)}}{F_{4(-52)}}$ | 2 | 26 |
| $J_3^{\mathbb{H}}$ | $\frac{SU^*(6)}{USp(6)}$ | 2 | 14 |
| $J_3^{\mathbb{C}}$ | $\frac{SL(3,\mathbb{C})}{SU(3)}$ | 2 | 8 |
| $J_3^{\mathbb{R}}$ | $\frac{SL(3,\mathbb{R})}{SO(3)}$ | 2 | 5 |

freedom which are not stabilized by the AM at the horizon of the extremal black hole. The general coset structure of such "moduli spaces" has the orbit stabilizer as global isometry, and its corresponding mcs as isotropy group; thus, by virtue of the treatment above (cf. (32)), one can generally write that

$$\mathcal{M}_{BPS}=\frac{H_0}{\mathrm{mcs}\,(H_0)}\simeq\varnothing\,, \tag{39}$$

$$\mathcal{M}_{nBPS,Z\neq0}=\frac{\widehat{H}}{\mathrm{mcs}\left(\widehat{H}\right)}=\frac{\widehat{H}}{\widehat{h}}\,,\quad \dim_{\mathbb{R}}\left(\mathcal{M}_{nBPS,Z\neq0}\right)=\sharp_{nBPS,Z\neq0}\,, \tag{40}$$

$$\mathcal{M}_{nBPS,Z=0}=\frac{\widetilde{H}}{\mathrm{mcs}\left(\widetilde{H}\right)}=\frac{\widetilde{H}}{\widetilde{h}}\,,\quad \dim_{\mathbb{R}}\left(\mathcal{M}_{nBPS,Z=0}\right)=\sharp_{nBPS,Z=0}\,, \tag{41}$$

where the non-existence of $\mathcal{M}_{BPS}$ follows from (25). This means that in $\mathcal{N}=2$ symmetric supergravities all critical points of $V_{BH}$ supported by "large" $U$-duality orbits are *stable*, up to a (possibly vanishing) certain number $\sharp$ of "flat" directions, spanning some proper subspace of the scalar manifold itself:

$$\mathcal{M}_{nBPS,Z\neq0}\subsetneq M_{n_V}\,, \tag{42}$$

$$\mathcal{M}_{nBPS,Z=0}\subsetneq M_{n_V}\,. \tag{43}$$

Tables 3 and 4 report spaces $\mathcal{M}_{nBPS,Z\neq0}$ and $\mathcal{M}_{nBPS,Z=0}$, respectively [55].

Interestingly, the "moduli space" $\mathcal{M}_{nBPS,Z\neq0}$ of non-BPS $Z_H\neq0$ attractors is the scalar manifold of the corresponding "parent" theory in $4+1$ space-time dimensions [55] (see also [57] and [58] for a result holding for generic special $d$-geometries).

Table 4: "Moduli spaces" of non-BPS $Z_H = 0$ extremal black hole attractors in $\mathcal{N} = 2$, $D = 4$ *symmetric* Maxwell-Einstein supergravities. They are (non-special) symmetric Kähler manifolds.

| | $\frac{\widetilde{H}}{mcs(\widetilde{H})} \equiv \frac{\widetilde{H}}{\widetilde{h}' \times U(1)}$ | $r$ | $\dim_{\mathbb{C}}$ |
|---|---|---|---|
| *minimal coupling* $n \in \mathbb{N}$ | $\frac{SU(1,n-1)}{U(1)\times SU(n-1)}$ | 1 | $n-1$ |
| $\mathbb{R} \oplus \mathbf{\Gamma}_{1,n-1}$, $n \in \mathbb{N}$ | $\frac{SO(2,n-2)}{SO(2)\times SO(n-2)}$, $n \geqslant 3$ | $1\ (n=3)$<br>$2\ (n \geqslant 4)$ | $n-2$ |
| $J_3^{\mathbb{O}}$ | $\frac{E_{6(-14)}}{SO(10)\times U(1)}$ | 2 | 16 |
| $J_3^{\mathbb{H}}$ | $\frac{SU(4,2)}{SU(4)\times SU(2)\times U(1)}$ | 2 | 8 |
| $J_3^{\mathbb{C}}$ | $\frac{SU(2,1)}{SU(2)\times U(1)} \times \frac{SU(1,2)}{SU(2)\times U(1)}$ | 2 | 4 |
| $J_3^{\mathbb{R}}$ | $\frac{SU(2,1)}{SU(2)\times U(1)}$ | 1 | 2 |

### 2.3 "Moduli spaces" of the ADM mass

Remarkably, by the thumb rule of orbit stabilizer modded by its mcs, one can associate "moduli spaces" also to "small" $U$-duality orbits, which do not attractor black holes: indeed, as mentioned above, the corresponding black hole has vanishing entropy in the Einstein, two-derivative approximation, and no AM (at least in the sense pointed out in the previous section; see [59]) holds [47, 60–62]. For "small" U-duality orbits there exists a "moduli space" also when the semi-simple part of the orbit stabilizer is a compact real Lie group: in such cases, the "moduli space" is spanned by the non-reductive, translational part of the orbit stabilizer itself [47, 62]. *Ça va sans dire* that for "small" orbits, there is no event horizon of the extremal black hole at which the N=2 central charge should be evaluated and no AM holds: in these cases, one may consider the asymptotical, spacial limit of the black hole, and put forward the interpretation of the "moduli spaces" associated to "small" orbits as "moduli spaces"of the ADM mass [9] of the "small" black hole itself.

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