

Generalisation of affine Lie algebras on compact real manifolds

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Abstract

We report on recent work concerning a new type of generalised Kac-Moody algebras based on the spaces of differentiable mappings from compact manifolds or homogeneous spaces onto compact Lie groups.



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1 Introduction

Among the infinite-dimensional groups and algebras motivated by physical problems, the Virasoro, Kac-Moody, current and W -algebras and their representations are the most relevant representatives, and constitute a fundamental tool in several theories, such as Conformal Field Theory, gauge and string theories or SUGRA models (see [1–3] and references therein). It turns out that Kac-Moody algebras, as well as the associated Virasoro algebras, provide a natural framework for the unification of symmetry and locality properties [4]. Basing on different physical assumptions, several generalisations of these algebraic structures have been proposed, usually from an analytic point of view, rather than on the axiomatic construction of these entities [5]. In this context, the quasisimple Lie algebras [6], generalised Kac-Moody algebras based on geometrical properties of closed surfaces [7] as well as several hierarchies of centrally extended algebras are worthy to be mentioned [8–11].

In most of these constructions, the one-dimensional sphere S^1 plays a relevant role, a fact that suggests that, for other physical models involving more than one degree of freedom and related to some basis manifold, a similar procedure can be proposed, provided that the manifold is either compact or presents some peculiar properties that guarantee convergence of integrals. This situation was the starting point for the general procedure initiated in [12],

where a systematic construction of generalised Kac-Moody algebras based on compact manifolds \mathcal{M} related to either a Lie group or an appropriate homogeneous space was proposed. Under these assumptions, harmonic functions on the manifold can be described in terms of the representation theory of the corresponding Lie group, allowing us, in particular, to identify a complete set of Hermitean labelling operators. An important difference of this generalisation with respect to the well-known class of usual Kac-Moody algebras and other generalisations resides in the fact that our construction, based on the Fourier expansion on compact manifolds, does not imply in general the existence of simple roots, even if a root structure can always be identified.

Besides the interest of these generalised Kac-Moody algebras from the mathematical point of view of, as this kind of algebras is naturally related to higher-dimensional compact manifolds, the question of their relevance in theories involving higher dimensional space-times such as Kaluza-Klein theories, supergravity, *etc* is of certainly of physical interest.

2 The algorithmic construction of generalised Kac-Moody algebras

The construction of generalised Kac-Moody algebras proposed in [12] for the case of manifolds associated to either a compact Lie group G_c or a coset space G_c/H (via the exponential map, see [12] for details) starts with a simple compact¹ Lie algebra \mathfrak{g} , a given basis $\{T_a, a = 1, \dots, \dim \mathfrak{g}\}$ with structure tensor

$$[T_a, T_b] = i f_{ab}{}^c T_c,$$

and Killing form

$$\langle T_a, T_b \rangle_0 = g_{ab} \equiv \text{Tr}(\text{ad}(T_a)\text{ad}(T_b)). \tag{1}$$

Denoting by V the volume elements of the associated compact $n = (p + q)$ -dimensional manifold \mathcal{M} (with $\mathcal{M} \simeq G_c$ or $\mathcal{M} \simeq G_c/H$, we consider a local coordinate frame $y^A = (\varphi^i, u^r)$ with $1 \leq i \leq p, 1 \leq r \leq q$, such that the condition

$$\int_{\mathcal{M}} d\mu(\mathcal{M}) = \frac{1}{V} \int_{\mathcal{M}} d^p \varphi d^q u = 1$$

holds. On \mathcal{M} we consider the set of square integrable functions periodic in all φ -directions, but non-periodic in the u -directions. The space $L^2(\mathcal{M})$ admits a complete orthonormal Hilbert basis

$$\mathcal{B} = \{\rho_I(\varphi, u), I \in \mathcal{I}\}, \tag{2}$$

with respect to the Hermitean scalar product on $L^2(\mathcal{M})$, where \mathcal{I} denotes a minimal (finite) set of labels required to identify the states unambiguously [13]. In these conditions, we define a space of smooth mappings from \mathcal{M} into \mathfrak{g} as

$$\mathfrak{g}(\mathcal{M}) = \{T_{aI} = T_a \rho_I(\varphi, u), a = 1, \dots, \dim \mathfrak{g}, I \in \mathcal{I}\}.$$

On this space, that inherits the structure of a Lie algebra, the Lie brackets are well defined and adopt the generic form

$$[T_{aI}, T_{bJ}] = i f_{ab}{}^c c_{IJ}{}^K T_{cK}, \tag{3}$$

¹This analysis can of course be extended to any simple (real or complex) Lie algebra. However, only in the case of compact Lie algebras, the representation theory has been analysed (see below).

where the coefficients c_{IJ}^K are those of the Fourier expansion of products of elements in the basis \mathcal{B} . For the case where the manifold \mathcal{M} is related to a compact Lie groups G_c , these can be associated to the Clebsch-Gordan coefficients of G_c . In particular, the Killing form in $\mathfrak{g}(\mathcal{M})$ is given by

$$\langle X, Y \rangle_1 = \int_{\mathcal{M}} d\mu(\mathcal{M}) \langle X, Y \rangle_0, \quad X, Y \in \mathfrak{g}(\mathcal{M}) \tag{4}$$

(with $\langle X, Y \rangle_0$ being the Killing form in (1)), from which the relations

$$\rho_I(\varphi, u) = \eta_{IJ} \bar{\rho}^J(\varphi, u), \quad \langle T_{aI}, T_{bJ} \rangle_1 = g_{ab} \eta_{IJ},$$

follow at once. The first relation simply means that $\bar{\rho}^J \in L^2(\mathcal{M})$, and thus extends in the basis \mathcal{B} given by (2).

In a second step, the existence of central extensions for the preceding algebras is analyzed. Following a general approach based on cohomological methods, the central extension is obtained through the 2-cocycle

$$\omega_c(X, Y) = \int_{\mathcal{M}} \langle X, \partial_i Y \, d\varphi^i + \partial_s Y \, du^s \rangle_0 \wedge \gamma, \tag{5}$$

with γ being a closed $(n - 1)$ -current associated to a closed loop C . In this context, it should be taken into account that central extensions are associated to compact one-dimensional submanifolds of \mathcal{M} , *i.e.* curves, and that the procedure cannot be extrapolated to maps from higher-dimensional manifolds onto \mathcal{M} [14]. Specifically, we consider

$$\gamma_{(A)} = (-1)^A k_A dy^1 \wedge \dots \wedge dy^{A-1} \wedge dy^{A+1} \wedge \dots \wedge dy^n, \quad A = 1, \dots, n, \quad k_A \in \mathbb{R}.$$

This leads to the identity

$$\omega_{(A)}(T_{aI}, T_{bJ}) = k_A g_{ab} \int_{\mathcal{M}} d\mu(\mathcal{M}) \rho_I(\varphi, u) \partial_A \rho_J(\varphi, u) = k_A g_{ab} d_{AIJ}, \tag{6}$$

hence for the centrally extended algebra $\mathfrak{g}(\mathcal{M})$ we get the commutator

$$[T_{aI}, T_{bJ}] = i f_{ab}{}^c c_{IJ}^K T_{cK} + g_{ab} \sum_{A=1}^n k_A d_{AIJ}. \tag{7}$$

It is not casual that this algebra has a deep similitude with the current algebra defined through

$$[T_a(y), T_{a'}(y')] = i f_{aa'}{}^b T_b(y) \delta^n(y - y') - i \sum_{A=1}^n k_A \partial_A \delta^n(y - y'), \tag{8}$$

and possessing Schwinger terms. Actually, centrally extended extensions of the generalised Kac-Moody algebras associated to the compact manifolds \mathbb{S}^2 and $\mathbb{S}^1 \times \mathbb{S}^1$ were determined in [15] by means of current algebras, showing the validity of the procedure.

In a third step, derivations ∂_A of the generalised Kac-Moody algebra $\mathfrak{g}(\mathcal{M})$ are considered. This is a technically delicate step, as the variables φ are periodic, whereas the variables u do not exhibit periodicity properties. In other words, the operators $d_j = -i\partial_{\varphi^j}$ are (commuting) Hermitean, while the operators $d_s = -i\partial_{u^s}$ are not Hermitean. In order to obtain a complete set of commuting Hermitean operators, we use the identification of the manifold \mathcal{M} with a

compact Lie group (coset space). To this extent, an embedding $\mathfrak{g}_c \subseteq \mathfrak{g}_m$ of \mathfrak{g}_c into a higher-rank Lie algebra \mathfrak{g}_m is used, with \mathfrak{g}_c the Lie algebra of G_c , and such that the basis functions of (2) belong to an irreducible unitary representation of \mathfrak{g}_m . The generators of the latter can be realised as Hermitean differential operators acting naturally on the manifold; in particular, the elements h_1, \dots, h_k of the Cartan subalgebra of \mathfrak{g}_m (where k is the rank of \mathfrak{g}_m), are realised as the Hermitean operators

$$h_j = -if_j^A(y)\partial_A, \quad 1 \leq j \leq k.$$

A particularity of these operators is that the boundary term associated to all u -directions vanishes. Among the operators $\{d_1, \dots, d_p, h_1, \dots, h_k\}$ we determine a maximal set of commuting operators

$$D_j = -if_j^A(y)\partial_A, \quad j = 1, \dots, r,$$

that satisfy the constraints

$$\partial_A f_j^A(y) = 0, \quad \text{and} \quad f_j^r| = 0, \quad j = 1, \dots, r, \tag{9}$$

related to Hermiticity. In these expressions, $f_j^r|$ represent the boundary terms associated to all u -directions that must vanish. Note further that when $\mathcal{M} = \mathbb{T}^n$, as all directions are periodic, we have $r = n$, but for a generic n -dimensional manifold \mathcal{M} we have $r < n$. It can be easily shown that the D_i act diagonally on the functions ρ_I , leading to an eigenvalue problem

$$D_j(\rho_I(y)) = I(j)\rho_I(y),$$

with $I(j)$ the corresponding eigenvalue.

The Hermitean operators D_j and central extensions of the generalised Kac-Moody algebra are deeply related through the closed $(n-1)$ -forms ($j = 1, \dots, r$)

$$\gamma_j = k_j \sum_{A=1}^n (-1)^A f_j^A(y) dy^1 \wedge \dots \wedge dy^{A-1} \wedge dy^{A+1} \wedge \dots \wedge dy^n, \quad j = 1, \dots, r, \quad k_j \in \mathbb{R}, \tag{10}$$

with corresponding 2-cocycles given by (see equation (6))

$$\omega_k(T_{aI}, T_{bJ}) = k_k J(k) g_{ab} \eta_{IJ}. \tag{11}$$

Summarising, the generalised Kac-Moody algebra $\widehat{\mathfrak{g}}(\mathcal{M})$ associated to the compact Lie algebra \mathfrak{g} and the compact manifold \mathcal{M} is determined by the following data

1. Generators T_{aI} belonging to $\mathfrak{g}(\mathcal{M})$;
2. Commuting Hermitean operators D_1, \dots, D_r ;
3. Central charges k_1, \dots, k_r associated to the Hermitean operators.

If $I(j)$ denotes the eigenvalue of D_j (see (10)), the non-vanishing brackets of the generalised Kac-Moody algebra associated to \mathcal{M} are

$$\begin{aligned} [T_{aI}, T_{bJ}] &= i f_{ab}{}^c c_{IJ}{}^K T_{cK} + g_{ab} \eta_{IJ} \sum_{j=1}^r k_j I(j), \\ [D_j, T_{aI}] &= I(j) T_{aI}, \end{aligned} \tag{12}$$

where $I(j)$ is the eigenvalue of D_j . Recall again that the central charges and the Hermitian operators are both associated to the closed $(n-1)$ -form given by (10).

As shown in [12], the choice of $G_c = U(1)^n$ leads to a generalised Kac-Moody algebra that structurally coincides with the generalised algebras based on the torus \mathbb{T}^n studied and analyzed in [6], and that actually correspond to specific cases of the wide class of so-called ‘quasi-simple Lie algebras’.

3 Identification of roots in $\widehat{\mathfrak{g}}(\mathcal{M})$

The fourth step is devoted to the identification of a root structure based on equation (12) associated to generalised Kac-Moody algebras. Supposed that the initial simple Lie algebra \mathfrak{g} has rank ℓ and let Σ denotes the root system with respect to the Cartan subalgebra $H^i, i = 1, \dots, \ell$, we consider the root operators $E_\alpha, \alpha \in \Sigma$ in the Cartan-Weyl basis. Defining

$$\widehat{\mathfrak{g}}(\mathcal{M}) = \text{Span}\{T_{aI}, D_j, k_j, a = 1, \dots, \dim \mathfrak{g}, I \in \mathcal{I}, j = 1, \dots, r\}, \tag{13}$$

the Cartan subalgebra of the latter is spanned by H^i, D_j and k_j ($i = 1, \dots, \ell, j = 1, \dots, r$). Taking the Cartan-Weyl basis H^i, E_{aI} and the Killing form as defined in (4), application of the procedure described in [16] shows that the Killing form of $\widehat{\mathfrak{g}}(\mathcal{M})$ satisfies

$$\begin{aligned} \langle T_{aI}, T_{bJ} \rangle &= \eta_{IJ} g_{ab}, & \langle D_j, T_{aI} \rangle &= \langle k_j, T_{aI} \rangle = 0, \\ \langle k_i, k_j \rangle &= \langle D_i, D_j \rangle = 0, & \langle D_i, k_j \rangle &= \delta_j^i. \end{aligned} \tag{14}$$

From this we get the (infinite-dimensional) root spaces (where $\mathbf{n} = (n_1, \dots, n_r)$)

$$\begin{aligned} \mathfrak{g}_{(\alpha, \mathbf{n})} &= \{E_{aI}, \text{ with } I(1) = n_1, \dots, I(r) = n_r\}, & \alpha \in \Sigma, & n_1, \dots, n_r \in \mathbb{Z}, \\ \mathfrak{g}_{(0, \mathbf{n})} &= \{H^i, \text{ with } I(1) = n_1, \dots, I(r) = n_r\}, & n_1, \dots, n_r \in \mathbb{Z}, \end{aligned} \tag{15}$$

with commutation relations

$$\begin{aligned} [\mathfrak{g}_{(0, \mathbf{n})}, \mathfrak{g}_{(\alpha, \mathbf{m})}] &\subset \mathfrak{g}_{(\alpha, \mathbf{m} + \mathbf{n})}, \\ [\mathfrak{g}_{(\alpha, \mathbf{m})}, \mathfrak{g}_{(\beta, \mathbf{n})}] &\subset \mathfrak{g}_{(\alpha + \beta, \mathbf{m} + \mathbf{n})}, \quad \alpha + \beta \in \Sigma. \end{aligned}$$

An important difference with respect to the usual Kac-Moody algebras is that, in this case, the commutator between elements depends also on the representation theory of G_c , specifically in connection with the Clebsch-Gordan coefficients c_{IJ}^K . This shows that the construction goes beyond the traditional root theory, as it also involves the so-called labelling problem for embedded algebras [13].

Explicit construction of these generalised structures was obtained in [12] for the case of manifolds isomorphic to the spheres \mathbb{S}^n , specifically for the values $n = 2$ with $SU(2)/U(1)$, $n = 3$ for $SU(2)$ and $SO(4)/SO(3)$, $n = 5$ for $SU(3)/SU(2)$ and $n = 6$ for $G_2/SU(3)$.

Concerning the representation theory of generalised Kac-Moody algebras, the case of the n -dimensional torus \mathbb{T}^n has been inspected in some detail in [12], corresponding to the Lie group $U(1)^n$. An extrapolation to other more complicated manifolds is a delicate task, the technical difficulties of which have not yet been solved satisfactorily. However, for the two-dimensional case and the manifolds \mathbb{T}^2 and \mathbb{S}^2 , an alternative ansatz has been proposed in [17] and [18], based on the observation that the Kac-Moody and the corresponding Virasoro algebras associated to these manifolds can be constructed naturally from the usual Kac-Moody and Virasoro algebras. More specifically, in this case we have assumed that the Laurent modes of the usual Kac-Moody and Virasoro algebras can themselves be (Fourier) developed in an adapted manner on the two-sphere and the two-torus, respectively [17]. This assumption enabled us to reproduce the generalised Kac-Moody algebras associated to \mathbb{S}^2 or \mathbb{T}^2 , in a semi-direct product with a subalgebra of vector fields of the two-torus and the two-sphere

$$\text{Vir}(\mathcal{M}) \ltimes \widehat{\mathfrak{g}}(\mathcal{M}), \quad \text{with } \mathcal{M} = \mathbb{S}^1, \text{ or } \mathbb{T}^2. \tag{16}$$

The algebras $\text{Vir}(\mathcal{M})$ can be seen as extensions of the Virasoro algebras in these cases. The interesting observation of this construction is that it leads naturally to central extensions. The fermions [17] and boson [18] realisations subsequently obtained lead automatically to a Fock space construction and thus to a unitary representation bounded from below. In the case of the bosonic construction, we have introduced vertex operators along the lines of the vertex operator in string theory [19, 20].

4 Concluding remarks and future prospects

We have reported on recent work concerning on the construction of generalised Kac-Moody algebras for the class of compact Lie groups and certain coset spaces determined by a closed subgroup, and the analysis of some of its main features that may be of interest in physical applications, such as the existence of a root system and central extensions. The procedure can formally be developed for any compact manifold or homogeneous space of the specified type, with the main difficulties being of computational nature. Whether this class of extensions fits naturally in the description of physical phenomenology, is a problem that has still to be explored in more detail.

The next natural step, besides specific applications, consists in proposing an analogous construction for the case where the basis manifold \mathcal{M} is no more compact. Some results in this direction actually exist, such as the work [9], but a general approach has not been formulated yet. Among the obstructions observed in this general frame, the acute divergence problems that arise in the integration theory of non-compact manifolds, as well as the technical difficulties emerging in the cohomological formulation of central extensions (see equation (5)), are the most relevant. Inspection of several examples suggest that additional techniques have to be considered to cover this case appropriately, in order to obtain a description the validity of which is not restricted to very particular manifolds. A successful approach in this sense could possibly be of interest in the context of M -theory or supergravity models, studying whether there is a connection between the central extensions of the generalised Kac-Moody algebra and super-membrane solutions in extended SUGRA.

The fermionic [17] and bosonic [18] construction obtained in the case of S^2 and T^2 can be easily extended to the n -tori T^n . This extension leads to a hierarchy of algebras (with the notations of (16))

$$\text{Vir}(T^n) \ltimes \widehat{\mathfrak{g}}(T^n) \subset \text{Vir}(T^{n-1}) \ltimes \widehat{\mathfrak{g}}(T^{n-1}) \subset \dots \subset \text{Vir} \ltimes \widehat{\mathfrak{g}}, \quad (17)$$

where at the last stage we have the usual Virasoro and Kac-Moody algebras. This series of embeddings could play a role in toroidal compactifications, a very important notion in higher dimensional supergravity.

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