SciPost Phys. Proc. 14, 035 (2023)

Vinberg's T-algebras: From exceptional periodicity to black hole entropy

Alessio Marrani⋆

Instituto de Física Teorica, Dep.to de Física, Universidad de Murcia,
Campus de Espinardo, E-30100, Spain

⋆ alessio.marrani@um.es

34th International Colloquium on Group Theoretical Methods in Physics
Strasbourg, 18-22 July 2022
doi:10.21468/SciPostPhysProc.14

Abstract

We introduce the so-called Magic Star (MS) projection within the root lattice of finite-dimensional exceptional Lie algebras, and relate it to rank-3 simple and semi-simple Jordan algebras. By relying on the Bott periodicity of reality and conjugacy properties of spinor representations, we present the so-called Exceptional Periodicity (EP) algebras, which are finite-dimensional algebras, violating the Jacobi identity, and providing an alternative with respect to Kac-Moody infinite-dimensional Lie algebras. Remarkably, also EP algebras can be characterized in terms of a MS projection, exploiting special Vinberg T-algebras, a class of generalized Hermitian matrix algebras introduced by Vinberg in the '60s within his theory of homogeneous convex cones. As physical applications, we highlight the role of the invariant norm of special Vinberg T-algebras in Maxwell-Einstein-scalar theories in 5 space-time dimensions, in which the Bekenstein-Hawking entropy of extremal black strings can be expressed in terms of the cubic polynomial norm of the T-algebras.

Received 24-06-2023
Accepted 29-08-2023
Published 24-11-2023
doi:10.21468/SciPostPhysProc.14.035

Contents

1 Projecting root lattices onto the magic star

Within the r-dimensional root lattice of $\mathfrak{g}_2$, $\mathfrak{f}_4$, $\mathfrak{e}_6$, $\mathfrak{e}_7$ and $\mathfrak{e}_8$ (with $r = 2, 4, 6, 7, 8$, resp.), one can find a plane (defined by the two Cartans of an $\mathfrak{a}_2$ subalgebra) on which the projection of the roots results into the so-called "Magic Star" (MS) (reported in Fig. 1). To the best of our knowledge, the MS was firstly observed in late '90s by Mukai[1] [2], and later re-discovered and treated in some detail by Truini [3] (see also [4]), within a different approach relying Jordan Pairs [5]; see also [1].

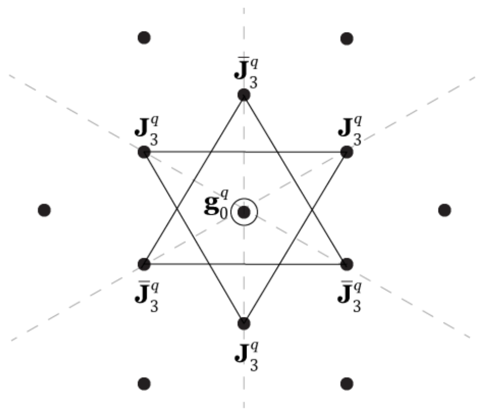

Figure 1: The Magic Star of exceptional Lie algebras [2, 3]. $\mathbf{J}_3^q$ denotes a rank-3 simple Jordan algebra, realized as matrix algebra of 3×3 Hermitian matrices over Hurwitz's division algebras $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ (of real dimension $q = \dim_{\mathbb{R}} \mathbb{A} = 1, 2, 4, 8$, resp.). The limit case of $\mathfrak{g}_2$ (corresponding to $q = -2/3$) corresponds to a trivial Jordan algebra, given by the identity element only: $\mathbf{J}_3^{-2/3} \equiv \mathbb{I} := diag(1, 1, 1)$.

The existence of the MS relies on the so-called (not necessarily maximal, generally non-symmetric) MS embedding/decomposition[2]

$$\mathfrak{qconf}\left(\mathbf{J}_3^q\right) \supset \mathfrak{a}_2 \oplus \mathfrak{str}_0\left(\mathbf{J}_3^q\right), \tag{1}$$

where $\mathfrak{qconf}\left(\mathbf{J}_3^q\right)$ and $\mathfrak{str}_0\left(\mathbf{J}_3^q\right)$ stand for the *quasi-conformal* resp. the *reduced structure* Lie algebra of $\mathbf{J}_3^q$ (see e.g. [13, 14] for basic definitions, and a list of Refs.).

Over $\mathbb{C}$, (1) implies [3, 4]

$$\mathfrak{qconf}\left(\mathbf{J}_3^q\right) = \mathfrak{a}_2 \oplus \mathfrak{str}_0\left(\mathbf{J}_3^q\right) \oplus \mathbf{3} \times \mathbf{J}_3^q \oplus \overline{\mathbf{3}} \times \overline{\mathbf{J}_3^q}. \tag{2}$$

Upon setting $q = 8, 4, 2, 1, 0, -2/3, -1$, one obtains the exceptional sequence (or exceptional series) Table 1, cf. e.g. [8].[3]

$\mathbf{J}_3^q$ stands for the rank-3 simple Jordan algebra [10] (cfr. e.g. [9], and Refs. therein) associated to the parameter q, which for $q = 8, 4, 2, 1$ is the real dimension of the division algebra $\mathbb{A}$ on which the corresponding Jordan algebra is realized as a 3×3 generalized matrix

[1]Mukai used the name "$\mathfrak{g}_2$ *decomposition*".

[2]For an application to supergravity, see [6] (where MS embedding was named Jordan pairs' embedding), as well as [7], in which the MS embedding was elucidated to be nothing but the $D = 5$ instance of the so-called super-Ehlers embedding.

[3]Note that we consider $\mathfrak{b}_3$, corresponding to $q = -1/3$ and absent in [8].

Table 1

q	8	4	2	1	0	$-1/3$	$-2/3$	-1
$\mathfrak{qconf}\left(\mathbf{J}_3^q\right)$	$\mathfrak{e}_8$	$\mathfrak{e}_7$	$\mathfrak{e}_6$	$\mathfrak{f}_4$	$\mathfrak{d}_4$	$\mathfrak{b}_3$	$\mathfrak{g}_2$	$\mathfrak{a}_2$
$\mathfrak{str}_0\left(\mathbf{J}_3^q\right)$	$\mathfrak{e}_6$	$\mathfrak{a}_5$	$\mathfrak{a}_2 \oplus \mathfrak{a}_2$	$\mathfrak{a}_2$	$\mathbb{C} \oplus \mathbb{C}$	$\mathbb{C}$	0	$-$

algebra with the property of $\mathbb{A}$-Hermiticity: $q = \dim_{\mathbb{R}} \mathbb{A} = 8, 4, 2, 1$ for $\mathbb{A} = \mathbb{O}, \mathbb{H}, \mathbb{C}, \mathbb{R}$, resp., and $\mathbf{J}_3^q \equiv \mathbf{J}_3^{\mathbb{A}} \equiv H_3(\mathbb{A})$ are equivalent notations. Remarkably, $\mathfrak{qconf}\left(\mathbf{J}_3^q\right)$ and $\mathfrak{str}_0\left(\mathbf{J}_3^q\right)$ span the entries of the fourth resp. second row/column of the Freudenthal-Tits Magic Square [11, 12] when setting $q = 8, 4, 2, 1$. From the classification of finite-dimensional, semi-simple cubic Jordan algebras [10], $\mathbf{J}_3^0 \equiv \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$ is the completely factorized (triality symmetric) rank-3 Jordan algebra, whereas $\mathbf{J}_3^{-1/3} \equiv \mathbb{C} \oplus \mathbb{C}$ and $\mathbf{J}_3^{-2/3} \equiv \mathbb{C}$ are its partial and total diagonal degenerations, respectively.

Within this report, we will consider things over $\mathbb{R}$. In this case, there are *at least* two non-compact real forms of the "enlarged" exceptional sequence $\left\{\mathfrak{qconf}\left(\mathbf{J}_3^q\right)\right\}_{q=8,4,2,1,0,-1/3,-2/3,-1}$ which can be easily interpreted in terms of symmetries of rank-3 real Jordan algebras: they are given in Tables 2 and Table 3. and they both pertain to the following non-compact, real form of (2)): $\mathfrak{qconf}\ \mathfrak{e}_8$

$$\mathfrak{qconf}\left(\mathbf{J}_3^q\right) = \mathfrak{sl}_{3,\mathbb{R}} \oplus \mathfrak{str}_0\left(\mathbf{J}_3^q\right) \oplus \mathbf{3} \times \mathbf{J}_3^q \oplus \mathbf{3}' \times \mathbf{J}_3^{q\prime}. \tag{3}$$

Table 2: The split real form of the exceptional sequence. In this case, for $q = 8, 4, 2, 1$, $\mathbf{J}_3^q \equiv \mathbf{J}_3^{\mathbb{A}_s} \equiv H_3(\mathbb{A}_s)$, where $\mathbb{A}_s$ is the split form of $\mathbb{A} = \mathbb{O}, \mathbb{H}, \mathbb{C}$, respectively.

q	8	4	2	1	0	$-1/3$	$-2/3$	-1
$\mathfrak{qconf}\left(\mathbf{J}_3^q\right)$	$\mathfrak{e}_{8(8)}$	$\mathfrak{e}_{7(7)}$	$\mathfrak{e}_{6(6)}$	$\mathfrak{f}_{4(4)}$	$\mathfrak{so}_{4,4}$	$\mathfrak{so}_{4,3}$	$\mathfrak{g}_{2(2)}$	$\mathfrak{sl}_{3,\mathbb{R}}$
$\mathfrak{str}_0(\mathbf{J}_3^q)$	$\mathfrak{e}_{6(6)}$	$\mathfrak{sl}_{6,\mathbb{R}}$	$\mathfrak{sl}_{3,\mathbb{R}} \oplus \mathfrak{sl}_{3,\mathbb{R}}$	$\mathfrak{sl}_{3,\mathbb{R}}$	$\mathbb{R} \oplus \mathbb{R}$	$\mathbb{R}$	0	$-$

Table 3: Another (non-split) non-compact real form of the exceptional sequence.

q	8	4	2	1	0	$-1/3$	$-2/3$	-1
$\mathfrak{qconf}\left(\mathbf{J}_3^q\right)$	$\mathfrak{e}_{8(-24)}$	$\mathfrak{e}_{7(-5)}$	$\mathfrak{e}_{6(2)}$	$\mathfrak{f}_{4(4)}$	$\mathfrak{so}_{4,4}$	$\mathfrak{so}_{4,3}$	$\mathfrak{g}_{2(2)}$	$\mathfrak{sl}_{3,\mathbb{R}}$
$\mathfrak{str}_0(\mathbf{J}_3^q)$	$\mathfrak{e}_{6(-26)}$	$\mathfrak{su}_6^*$	$(\mathfrak{sl}_{3,\mathbb{C}})_{\mathbb{R}}$	$\mathfrak{sl}_{3,\mathbb{R}}$	$\mathbb{R} \oplus \mathbb{R}$	$\mathbb{R}$	0	$-$

2 Spinor content of exceptional Lie algebras and Fierz identities in $8+q$ dimensions

The following maximal, Jordan algebraic embeddings

$$\begin{aligned} \mathbf{J}_3^{\mathbb{A}} &\supset \mathbb{R} \oplus \mathbf{J}_2^{\mathbb{A}}, \\ \mathbf{J}_3^{\mathbb{A}_s} &\supset \mathbb{R} \oplus \mathbf{J}_2^{\mathbb{A}_s}, \end{aligned} \tag{4}$$

enjoy the following matrix realization as ($r_i \in \mathbb{R}$, $A_i \in \mathbb{A}$ or $\mathbb{A}_s$, $i = 1, 2, 3$)

$$\mathbf{J}_3^{\mathbb{A}} \ni J = \begin{pmatrix} r_1 & A_1 & \overline{A}_2 \\ \overline{A}_1 & r_2 & A_3 \\ A_2 & \overline{A}_3 & r_3 \end{pmatrix} \Rightarrow J' = \begin{pmatrix} r_1 & A_1 & 0 \\ \overline{A}_1 & r_2 & 0 \\ 0 & 0 & r_3 \end{pmatrix} \in \mathbb{R} \oplus \mathbf{J}_2^{\mathbb{A}}, \tag{5}$$

where the bar denotes the conjugation in $\mathbb{A}$ or in $\mathbb{A}_s$. Usually, the matrix elements r_1 and r_2 are associated to lightcone degrees of freedom, i.e.

$$r_1 := x_+ + x_-, \; r_2 := x_+ - x_- . \tag{6}$$

Furthermore, the following algebraic isomorphisms hold (cf. e.g. [15]):

$$\mathbf{J}_2^{\mathbb{A}} \sim \mathbf{\Gamma}_{1,q+1}, \tag{7}$$
$$\mathbf{J}_2^{\mathbb{A}_s} \sim \mathbf{\Gamma}_{q/2+1,q/2+1}, \tag{8}$$

where $\mathbf{\Gamma}_{1,q+1}$ and $\mathbf{\Gamma}_{q/2+1,q/2+1}$ are (generally simple) Jordan algebras of rank 2 with a quadratic form of (Lorentian resp. Kleinian) signature $(1, q+1)$ resp. $(q/2+1, q/2+1)$, i.e. the Clifford algebras of $O(1, q+1)$ resp. $O(q/2+1, q/2+1)$; for this reason, it is customary to refer to (4) as to the the *spin-factor* embeddings.

By setting $\mathbb{A} = \mathbb{O}$, i.e. $q = 8$, in (4), and considering the various symmetries of Jordan algebras, one obtains the graded structure of suitable real forms of finite-dimensional exceptional Lie algebras with respect to the corresponding pseudo-orthogonal Lie algebras, thus obtaining the spinor content of the exceptional algebras themselves:

1. For what concerns the derivations $\mathfrak{der}$ (namely, the Lie algebra of the automorphism group) of the rank-3 Jordan algebras, one obtains the 2-graded structure of the real, compact form of $\mathfrak{f}_4$, namely:

$$\mathfrak{der}\left(\mathbf{J}_3^{\mathbb{O}}\right) \supset^{m,s} \mathfrak{der}\left(\mathbb{R} \oplus \mathbf{J}_2^{\mathbb{O}}\right) \Leftrightarrow \begin{cases} \mathfrak{f}_{4(-52)} \supset^{m,s} \mathfrak{so}_9, \\ \\ \mathfrak{f}_{4(-52)} = \mathfrak{so}_9 \oplus \mathbf{16}, \end{cases} \tag{9}$$

where $\mathbf{16}$ is the Majorana spinor irrepr. of $\mathbf{so}_9$, and the upperscripts "m" and "s" respectively indicate maximality and symmetric nature. The fact that the 2-graded vector space $\mathfrak{so}_9 \oplus \mathbf{16}$ can be endowed with the structure of a (simple, exceptional) Lie algebra, and thus satisfies the Jacobi identity (in particular, for three elements in $\mathbf{16}$), relies on a remarkable Fierz identity for $\mathfrak{so}_9$ gamma matrices.

2. At the level of the *reduced structure* Lie algebra $\mathfrak{str}_0$, one obtains the 3-graded structure of the real, minimally non-compact form of $\mathfrak{e}_6$, namely:

$$\mathfrak{str}_0\left(\mathbf{J}_3^{\mathbb{O}}\right) \supset^{m,s} \mathfrak{str}_0\left(\mathbb{R} \oplus \mathbf{J}_2^{\mathbb{O}}\right) \Leftrightarrow \begin{cases} \mathfrak{e}_{6(-26)} \supset^{m,s} \mathfrak{so}_{9,1} \oplus \mathbb{R}, \\ \\ \mathfrak{e}_{6(-26)} = \mathbf{16}'_{-1} \oplus \left(\mathfrak{so}_{9,1} \oplus \mathbb{R}\right)_0 \oplus \mathbf{16}_1, \\ \text{or} \\ \mathfrak{e}_{6(-26)} = \mathbf{16}_{-1} \oplus \left(\mathfrak{so}_{9,1} \oplus \mathbb{R}\right)_0 \oplus \mathbf{16}'_1, \end{cases} \tag{10}$$

where $\mathbf{16}$ and $\mathbf{16}'$ are the Majorana-Weyl (MW) spinors of $\mathbf{so}_{9,1}$, which constitute an example of *Jordan pair* which is not a pair of Jordan algebras (see e.g. [5], as well as [3, 4] for a recent treatment); also, the indeterminacy denoted by "or" depends on the *spinor polarization* of the embedding [16]. The fact that the 3-graded vector space(s) in the r.h.s. of (10) can be endowed with the structure of a (simple, exceptional) Lie algebra, and thus satisfies the Jacobi identity (in particular, for three elements in $\mathbf{16} \oplus \mathbf{16}'$), relies on a remarkable Fierz identity for $\mathfrak{so}_{9,1}$ gamma matrices. Note that $\mathfrak{str}\left(\mathbf{J}_3^{\mathbb{O}}\right) \simeq \mathfrak{str}_0\left(\mathbf{J}_3^{\mathbb{O}}\right) \oplus \mathbb{R}$ is isomorphic to the Lie algebra of the automorphism group $\mathrm{Aut}\left(\mathbf{J}_3^{\mathbb{O}}, \mathbf{J}_3^{\mathbb{O}\prime}\right)$ of the *Jordan pair* $\left(\mathbf{J}_3^{\mathbb{O}}, \mathbf{J}_3^{\mathbb{O}\prime}\right)$:

$$\mathfrak{str}\left(\mathbf{J}_3^{\mathbb{O}}\right) \simeq \mathrm{Lie}\left(\mathrm{Aut}\left(\left(\mathbf{J}_3^{\mathbb{O}}, \mathbf{J}_3^{\mathbb{O}\prime}\right)\right)\right) \simeq \mathfrak{der}\left(\mathbf{J}_3^{\mathbb{O}}, \mathbf{J}_3^{\mathbb{O}\prime}\right). \tag{11}$$

3. At the level of the *conformal* Lie algebra $\mathfrak{conf}$, one obtains

$$\mathfrak{conf}\left(\mathbf{J}_3^{\mathbb{O}}\right) \supset^{m,s} \mathfrak{conf}\left(\mathbb{R} \oplus \mathbf{J}_2^{\mathbb{O}}\right) \Leftrightarrow \begin{cases} \mathfrak{e}_{7(-25)} \supset^{m,s} \mathfrak{so}_{10,2} \oplus \mathfrak{sl}_{2,\mathbb{R}}\,, \\ \\ \mathfrak{e}_{7(-25)} = \mathfrak{so}_{10,2} \oplus \mathfrak{sl}_{2,\mathbb{R}} \oplus \left(\mathbf{32}^{(\prime)}, \mathbf{2}\right), \end{cases} \tag{12}$$

where $\mathbf{32}$ is the MW spinor of $\mathfrak{so}_{10,2}$, and the possible priming (denoting spinor conjugation) depends on the choice of the spinor polarization [16]. By further branching the $\mathfrak{sl}_{2,\mathbb{R}}$, one obtain a 5-grading of contact type (recently reconsidered within the classification worked out in [17]) of the real, minimally non-compact form of $\mathfrak{e}_7$, namely:

$$\begin{aligned} &\mathfrak{e}_{7(-25)} \supset \mathfrak{so}_{10,2} \oplus \mathbb{R}\,, \\ &\mathfrak{e}_{7(-25)} = \mathbf{1}_{-2} \oplus \mathbf{32}_{-1}^{(\prime)} \oplus \left(\mathfrak{so}_{10,2} \oplus \mathbb{R}\right)_0 \oplus \mathbf{32}_1^{(\prime)} \oplus \mathbf{1}_2. \end{aligned} \tag{13}$$

The fact that the 5-graded vector space(s) in the r.h.s. of (13) can be endowed with the structure of a (simple, exceptional) Lie algebra, and thus satisfies the Jacobi identity (in particular, for three elements in $\mathbf{32}^{(\prime)} \oplus \mathbf{32}^{(\prime)}$), relies on a remarkable Fierz identity for $\mathfrak{so}_{10,2}$ gamma matrices. Note that $\mathfrak{conf}\left(\mathbf{J}_3^{\mathbb{O}}\right)$ is isomorphic to the Lie algebra of the automorphism group $\mathrm{Aut}\left(\mathfrak{F}\left(\mathbf{J}_3^{\mathbb{O}}\right)\right)$ of the reduced Freudenthal triple system constructed over $\mathbf{J}_3^{\mathbb{O}}$:

$$\mathfrak{conf}\left(\mathbf{J}_3^{\mathbb{O}}\right) \simeq \mathrm{Lie}\left(\mathrm{Aut}\left(\mathfrak{F}\left(\mathbf{J}_3^{\mathbb{O}}\right)\right)\right) \simeq \mathfrak{der}\left(\mathfrak{F}\left(\mathbf{J}_3^{\mathbb{O}}\right)\right). \tag{14}$$

4. Finally, at the level of the *quasi-conformal* Lie algebra[4] $\mathfrak{qconf}$ [13, 14], one obtains the 2-graded structure of the real, minimally non-compact form of $\mathfrak{e}_8$, namely:

$$\mathfrak{qconf}\left(\mathbf{J}_3^{\mathbb{O}}\right) \supset^{m,s} \mathfrak{qconf}\left(\mathbb{R} \oplus \mathbf{J}_2^{\mathbb{O}}\right) \Leftrightarrow \begin{cases} \mathfrak{e}_{8(-24)} \supset^{m,s} \mathfrak{so}_{12,4}\,, \\ \\ \mathfrak{e}_{8(-24)} = \mathfrak{so}_{12,4} \oplus \mathbf{128}^{(\prime)}, \end{cases} \tag{15}$$

where $\mathbf{128}$ is the MW spinor of $\mathfrak{so}_{12,4}$, and, again, the possible priming (standing for spinorial conjugation) relates to the choice of the spinor polarization [16]. Further decomposition of $\mathfrak{so}_{12,4}$ yields to a 5-grading of "extended Poincaré" type [17]:

$$\begin{aligned} &\mathfrak{e}_{8(-24)} \supset \mathfrak{so}_{11,3} \oplus \mathbb{R}\,, \\ &\mathfrak{e}_{8(-24)} = \begin{cases} \mathbf{14}_{-2} \oplus \mathbf{64}'_{-1} \oplus \left(\mathfrak{so}_{11,3} \oplus \mathbb{R}\right)_0 \oplus \mathbf{64}_1 \oplus \mathbf{14}_2\,, \\ \qquad\qquad\text{or} \\ \mathbf{14}_{-2} \oplus \mathbf{64}_{-1} \oplus \left(\mathfrak{so}_{11,3} \oplus \mathbb{R}\right)_0 \oplus \mathbf{64}'_1 \oplus \mathbf{14}_2\,, \end{cases} \end{aligned} \tag{16}$$

where $\mathbf{64}$ is the MW spinor of $\mathfrak{so}_{11,3}$ and the "or" indeterminacy depends on the spinor polarization [16]. The fact that the 2-graded vector space $\mathfrak{so}_{12,4} \oplus \mathbf{128}^{(\prime)}$ can be endowed with the structure of a (simple,exceptional) Lie algebra, and thus satisfies the Jacobi identity (in particular, for three elements in $\mathbf{128}^{(\prime)}$), relies on a remarkable Fierz identity for $\mathfrak{so}_{12,4}$ gamma matrices. Equivalently, the fact that the 5-graded vector space(s) in the r.h.s. of (16) can be endowed with the structure of a (simple, exceptional) Lie algebra, and thus satisfies the Jacobi identity (in particular, for three elements in $\mathbf{64} \oplus \mathbf{64}'$), relies on a remarkable Fierz identity for $\mathfrak{so}_{11,3}$ gamma matrices.

[4] We recall that the quasi-conformal realization of $\mathfrak{e}_{8(-24)}$ concerns a non-linear action on an *extended* derived Freudenthal triple system $\mathfrak{EF}\left(\mathbf{J}_3^{\mathbb{O}}\right) \simeq \mathbb{R} \oplus \mathfrak{F}\left(\mathbf{J}_3^{\mathbb{O}}\right)$ [13].

3 From Bott periodicity to exceptional periodicity

Thus, we have related the existence of (finite-dimensional, simple) exceptional Lie algebras to some remarkable Fierz identities holding in $q+8$ dimensions (in particular, with signature $9+0$, $9+1$, $10+2$,and $12+4$, for $q=1,2,4$ and 8, respectively).

Now, by observing that the reality properties of spinors and the existence and symmetry of invariant spinor bilinears are periodic *mod* 8 (Bott periodicity), one can define some algebras which (for the moment, formally) generalize the spinor content of the real forms of exceptional Lie algebras discussed above: these are the so-called *"Exceptional Periodicity"* (EP) *algebras* [1, 18], and, as vector spaces, they are defined as follows ($n \in \mathbb{N} \cup \{0\}$ throughout[5]):

1. Level $\mathfrak{der}$:
$$\mathfrak{f}^{(n)}_{4(-52)} := \mathfrak{so}_{9+8n} \oplus \psi_{\mathbf{so}_{9+8n}}\,, \tag{17}$$
where $\psi_{\mathbf{so}_{9+8n}} \equiv \mathbf{2}^{4+4n}$ is the Majorana spinor of $\mathfrak{so}_{9+8n}$.

2. Level $\mathfrak{str}_0$:
$$\mathfrak{e}^{(n)}_{6(-26)} := \psi'_{\mathbf{so}_{9+8n,1},-1} \oplus \left(\mathfrak{so}_{9+8n,1} \oplus \mathbb{R}\right)_0 \oplus \psi_{\mathbf{so}_{9+8n,1},1}\,, \tag{18}$$
where $\psi_{\mathbf{so}_{9+8n,1}} \equiv \mathbf{2}^{4+4n}$ is the MW spinor of $\mathfrak{so}_{9+8n,1}$.

3. Level $\mathfrak{conf}$:
$$\begin{aligned}\mathfrak{e}^{(n)}_{7(-25)} :&= \left(\mathfrak{so}_{10+8n,2} \oplus \mathfrak{sl}_{2,\mathbb{R}}\right) \oplus \left(\psi_{\mathbf{so}_{10+8n,2}}, \mathbf{2}\right) \\ &= \mathbf{1}_{-2} \oplus \psi_{\mathbf{so}_{10+8n,2},-1} \oplus \left(\mathfrak{so}_{10+8n,2} \oplus \mathbb{R}\right)_0 \oplus \psi_{\mathbf{so}_{10+8n,2},1} \oplus \mathbf{1}_2\,,\end{aligned} \tag{19}$$
where $\psi_{\mathbf{so}_{10+8n,2}} \equiv \mathbf{2}^{5+4n}$ is the MW spinor of $\mathfrak{so}_{10+8n,2}$.

4. Level $\mathfrak{qconf}$:
$$\begin{aligned}\mathfrak{e}^{(n)}_{8(-24)} :&= \mathfrak{so}_{12+8n,4} \oplus \psi_{\mathbf{so}_{12+8n,4}} \\ &= (\mathbf{14+8n})_{-2} \oplus \psi'_{\mathbf{so}_{11+8n,3},-1} \oplus \left(\mathfrak{so}_{11+8n,3} \oplus \mathbb{R}\right)_0 \oplus \psi_{\mathbf{so}_{11+8n,3},1} \oplus (\mathbf{14+8n})_2\,,\end{aligned} \tag{20}$$
where $\psi_{\mathbf{so}_{12+8n,4}} \equiv \mathbf{2}^{7+4n}$ and $\psi_{\mathbf{so}_{11+8n,3}} \equiv \mathbf{2}^{6+4n}$ respectively denote the MW spinors of $\mathfrak{so}_{12+8n,4}$ and of $\mathfrak{so}_{11+8n,3}$.

A rigorous algebraic definition of the above EP algebras has been given in [18] (see also [1]) by introducing the notion of *generalized* roots, and by defining the structure constants in terms of (a suitably generalized) Kac's *asymmetry function* [19, 20]. In this report, we confine ourselves to remark that EP algebras are *not* simply non-reductive nor semisimple, spinor-affine extensions of (pseudo-)orthogonal Lie algebras, but their spinor generators are non-translational (i.e., non-Abelian), as are the spinor generators of[6] $\mathfrak{f}_{4(-52)} \equiv \mathfrak{f}^{(n=0)}_{4(-52)}$, $\mathfrak{e}_{6(-26)} \equiv \mathfrak{e}^{(n=0)}_{6(-26)}$, $\mathfrak{e}_{7(-25)} \equiv \mathfrak{e}^{(n=0)}_{7(-25)}$, and $\mathfrak{e}_{8(-24)} \equiv \mathfrak{e}^{(n=0)}_{8(-24)}$. This yields to the *violation* of the Jacobi identity when considering three spinor generators as input in the Jacobiator [18]. As of today, a rigorous, axiomatic treatment of EP algebras is missing: can EP algebras be defined in terms of some characterizing identities, going beyond Jacobi? This remains an open problem.

[5] Note that there has been a shift of unity with respect to the notation of [1] and [18]: the index n used here is actually $n-1$ of such Refs.

[6] The treatment on $\mathbb{R}$ given here is based on the EP generalization of the various symmetry Lie algebras of the Albert algebra $\mathbf{J}_3^{\mathbb{O}}$, and it yielded to some specific real forms of $\mathfrak{f}^{(n)}_4$, $\mathfrak{e}^{(n)}_6$, $\mathfrak{e}^{(n)}_7$ and $\mathfrak{e}^{(n)}_8$. Starting from $\mathbb{C}$, a rigorous definition of all real forms of EP algebras, by means of the introduction of suitable involutive morphisms within the corresponding EP generalized root lattices [18], will be the object of forthcoming works.

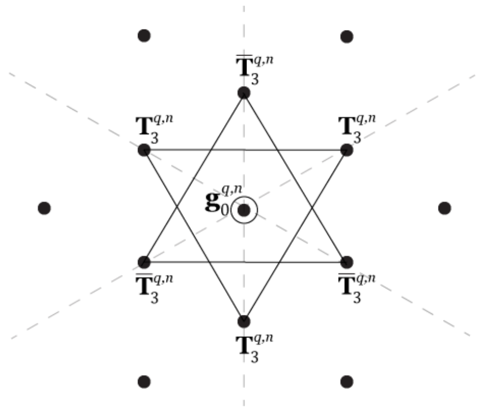

Figure 2: The *Magic Star* structure of the $\mathfrak{a}_2$-projection of the generalized root lattices of EP algebras. finite-dimensional [18]. $\mathbf{T}_3^{q,n}$ stands for a Vinberg T-algebra of rank-3 and of *special type* [22], parametrized by $q = 1, 2, 4, 8$ and $n \in N \cup \{0\}$, corresponding to $\mathfrak{f}_4^{(n)}$, $\mathfrak{e}_6^{(n)}$, $\mathfrak{e}_7^{(n)}$, $\mathfrak{e}_8^{(n)}$, respectively.

The crucial result, which motivates and renders all the above construction and the corresponding construction in the EP lattices non-trivial, is the following [18]: for $n > 0$, all EP algebras admit a $\mathfrak{a}_2$ subalgebra, such that the projection of their generalized root lattices onto the 2-dimensional plane defined by the Cartans of such $\mathfrak{a}_2$ has a *Magic Star structure*, with those generalized roots corresponding to the degeneracies on the tips of such EP-generalized Magic Star which can be endowed with an algebraic structure, denoted by $\mathbf{T}_3^{q,n}$, generalizing the rank-3 simple Jordan algebras $\mathbf{J}_3^q \equiv \mathbf{J}_3^{\mathbb{A}} \equiv H_3(\mathbb{A})$ mentioned above. The resulting, EP-generalized Magic Star is depicted in Fig. 2. Remarkably, such a generalization is[7] the *unique* possible one, and it is provided by the Hermitian part of (a class of) rank-3 T-algebras *of special type*. Such algebras were introduced some time ago by Vinberg [22], and they recently appeared in [23–25], in which they have been named *Vinberg special T-algebras*.

4 Vinberg special T-algebras and Bekenstein-Hawking entropy

The real forms of EP algebras resulting from the treatment given above, i.e. $\mathfrak{f}_{4(-52)}^{(n)}$, $\mathfrak{e}_{6(-26)}^{(n)}$, $\mathfrak{e}_{7(-25)}^{(n)}$, and $\mathfrak{e}_{8(-24)}^{(n)}$ (corresponding to $\mathfrak{der}$, $\mathfrak{str}_0$, $\mathfrak{conf}$ and $\mathfrak{qconf}$ levels, or, equivalently - by the symmetry of the Freudenthal-Tits Magic Square [11, 12] - to $q = 1, 2, 4$ and 8, respectively), the 3×3 generalized matrix algebras $\mathbf{T}_3^{q,n}$ corresponding to the set of generalized roots degenerating to a point on each of the tips of the EP-generalized Magic Star (depicted in Fig. 2) can be realized as follows:

$$
\mathbf{T}_3^{q,n} := \begin{pmatrix} r_1 & \mathbf{V}_{\mathfrak{so}_{q+8n}} & \psi_{\mathfrak{so}_{q+8n}} \\ \overline{\mathbf{V}}_{\mathfrak{so}_{q+8n}} & r_2 & \psi'_{\mathfrak{so}_{q+8n}} \\ \overline{\psi}_{\mathfrak{so}_{q+8n}} & \overline{\psi'}_{\mathfrak{so}_{q+8n}} & r_3 \end{pmatrix}, \tag{21}
$$

[7] Within a set of reasonable and intuitive assumptions [22].

where[8]

$$\mathbf{V}_{\mathfrak{so}_{q+8n}} := (\boldsymbol{q+8n}, \mathbf{1}) , \tag{22}$$
$$\psi_{\mathfrak{so}_{q+8n}} := \left(\mathbf{2}^{[(q+1)/2]+4n-1+\delta_{q,1}}, \mathbf{Fund}\left(\mathcal{S}_q\right)\right) , \tag{23}$$

are irreducible representation spaces of the Lie algebra

$$\mathfrak{so}_{q+8n} \oplus \mathcal{S}_q , \tag{24}$$

with

$$\mathcal{S}_q := \mathfrak{tri}_{\mathbb{A}} \ominus \mathfrak{so}_{\mathbb{A}} = 0, \mathfrak{so}_2, \mathfrak{su}_2, 0 , \quad \text{for } q = 1,2,4,8 \quad (\text{i.e., for } \mathbb{R},\mathbb{C},\mathbb{H},\mathbb{O}, \text{ resp.}) , \tag{25}$$

denoting the coset algebra of the *triality* symmetry $\mathbf{tri}_{\mathbb{A}}$ of $\mathbb{A}$ [26]:

$$\mathfrak{tri}_{\mathbb{A}} := \{(A,B,C) \,|A(xy) = B(x)y + xC(y),\ A,B,C \in \mathfrak{so}_{\mathbb{A}},\ x,y \in \mathbb{A}\} \tag{26}$$
$$= 0, \mathfrak{so}_2^{\oplus 2}, \mathfrak{so}_3^{\oplus 3}, \mathfrak{so}_8 , \quad \text{for } \mathbb{A} = \mathbb{R},\mathbb{C},\mathbb{H},\mathbb{O} , \tag{27}$$

modded by the *norm-preserving* symmetry $\mathfrak{so}_{\mathbb{A}}$of $\mathbb{A}$:

$$\mathfrak{so}_{\mathbb{A}} := \mathfrak{so}_q = 0, \mathfrak{so}_2, \mathfrak{so}_4, \mathfrak{so}_8 , \quad \text{for } \mathbb{A} = \mathbb{R},\mathbb{C},\mathbb{H},\mathbb{O} . \tag{28}$$

Actually, $\mathcal{S}_q$ is related to the reality properties of the spinors of $\mathfrak{so}_{q+8n}$, and in Physics it is named $\mathcal{R}$-symmetry. Furthermore, $\mathbf{Fund}\left(\mathcal{S}_q\right)$ denotes the smallest non-trivial representation of the simple Lie algebra $\mathcal{S}_q$ (if any):

$$\mathbf{Fund}\left(\mathcal{S}_q\right) = -, \mathbf{2}, \mathbf{2}, - , \quad \text{for } q = 1,2,4,8 , \tag{29}$$

with real dimension

$$\mathrm{fund}_q := \dim_{\mathbb{R}} \mathbf{Fund}\left(\mathcal{S}_q\right) = 1,2,2,1 , \quad \text{for } q = 1,2,4,8 . \tag{30}$$

Thus, the total real dimension of $\mathbf{T}_3^{q,n}$ is

$$\dim_{\mathbb{R}}(\mathbf{T}_3^{q,n}) = q + 3 + 8n + \mathrm{fund}_q \cdot 2^{[(q+1)/2]+4n+\delta_{q,1}} . \tag{31}$$

As mentioned above, $\mathbf{T}_3^{q,n}$ (21) is the Hermitian part of a certain class of generalized matrix algebras going under the name of rank-3 T-algebras, introduced sometime ago by Vinberg as a unique,consistent generalization of rank-3, simple Jordan algebras, within its theory of homogeneous convex cones [22]: more precisely, $\mathbf{T}_3^{q,n}$ has been dubbed *exceptional* T-algebra in Sec. 4.3 of [1]. Upon a slight generalization (i.e., by including $P + \dot{P}$ copies of spinor irreprs., and correspondingly extending $\mathcal{S}_q$ to the "full-fledged" $\mathcal{R}$-symmetry $\mathcal{S}_q\left(P, \dot{P}\right)$), $\mathbf{T}_3^{q,n}$ gets generalized to $\mathbf{T}_3^{q,n,P,\dot{P}}$ (with $P, \dot{P} \in \mathbb{N} \cup \{0\}$), which occur in the study of so-called homogeneous *real special* manifolds.[9] These are non-compact Riemannian spaces occurring as (vector multiplets') scalar manifolds of $\mathcal{N} = 2$-extended Maxwell-Einstein supergravity theories in $D = 4+1$ space-time dimensions, firstly discussed to some extent by Cecotti [28]. More recently, $\mathbf{T}_3^{q,n,P,\dot{P}}$ have appeared under the name of *Vinberg special T- algebras* in works on Vinberg's theory of homogeneous cones (and generalizations thereof) and on its relation to the entropy of extremal black holes in $\mathcal{N} = 2$-extended Maxwell-Einstein supergravity theories in $D = 3 + 1$ space-time dimensions [23–25].

[8] $[\cdot]$ denotes the integer part throughout.

[9] And, of course, in their images under R-map and c-map (cfr. e.g. [27], and Refs. therein).

The *unique* invariant structure of the algebra[10] $\mathbf{T}_3^{q,n} \equiv \mathbf{T}_3^{q,n,P,\dot{P}}\Big|_{P=1,\dot{P}=0}$ given by (21) is provided by its (formal) "determinant". In order to define it, let us introduce ($\mu = 0, 1, ..., q + 1 + 8n$)

$$V^{\mu} := \left(r_1, r_2, \mathbf{V}_{\mathbf{so}_{q+8n}}\right), \tag{32}$$

which, by recalling (6), is recognized to be a vector module of $Spin(q+1+8n, 1)$; we also denote the corresponding spinor of $\mathfrak{so}_{q+1+8n,1}$ (which is chiral for $q = 2, 4, 8$), of real dimension $\text{fund}_q \cdot 2^{[(q+1)/2]+4n+\delta_{q,1}}$, by $\Psi^{\alpha A}$ (where $\alpha = 1, ..., 2^{[(q+1)/2]+4n+\delta_{q,1}}$ and $A = 1, .., \text{fund}_q$). Then, the "determinant" of the generalized Hermitian matrix algebra $\mathbf{T}_3^{q,n}$, which defines the cubic norm $\mathbf{N}$ of $\mathbf{T}_3^{q,n}$ itself, is defined as

$$\mathbf{N}\left(\mathbf{T}_3^{q,n}\right) := \frac{1}{2}\eta_{\mu\nu}\left[r_3 V^{\mu} V^{\nu} + \gamma^{\mu}_{\alpha\beta}\Psi^{\alpha A}\Psi^{\beta}_{A} V^{\nu}\right], \tag{33}$$

where $\eta_{\mu\nu}$ is the symmetric bilinear invariant of the vector module V (32) of $\text{Spin}(q+1+8n, 1)$, and $\gamma^{\mu}_{\alpha\beta}$ are the gamma matrices of $\mathfrak{so}_{q+1+8n,1}$.

Remarkably, Ferrar's classification [29] of elements of a rank-3 Jordan algebras in terms of *invariant rank*$= 0, 1, 2, 3$ can be generalized to the classification of the elements of $\mathbf{T}_3^{q,n}$ depending on their invariant rank as well, defined as follows [18]:

$$\begin{aligned} \text{rank-3}: \quad & \mathbf{N} \neq 0, \\ \text{rank-2}: \quad & \mathbf{N} = 0, \\ \text{rank-1}: \quad & \partial\mathbf{N} = 0. \end{aligned} \tag{34}$$

In those (ungauged) $\mathcal{N} = 2$-extended Maxwell-Einstein supergravity theories in $D = 4 + 1$ space-time dimensions based on $\mathbf{T}_3^{q,n}$ [28], the magnetic charges of extremal black strings (with near-horizon geometry $AdS_3 \otimes S^2$) fit into $\mathbf{T}_3^{q,n}$ itself, and its Bekenstein-Hawking entropy S_{BS} enjoys the interestingly simple expression

$$S_{BS} = \pi\sqrt{|\mathbf{N}|}. \tag{35}$$

We conclude this report by pointing out that the entropy of the extremal dyonic black holes in the corresponding (ungauged) $(3+1)$-dimensional supergravity theory (obtained by compactifying the fourth spacial dimension on S^1 and keeping the massless sector) has been recently discussed in [24]. Analogue formulæ hold when considering the most general case $\mathbf{T}_3^{q,n,P,\dot{P}}$ (with $P, \dot{P} \in \mathbb{N} \cup \{0\}$).

References

[1] P. Truini, A. Marrani and M. Rios, *Magic star and exceptional periodicity: An approach to quantum gravity*, J. Phys.: Conf. Ser. **1194**, 012106 (2019), doi:10.1088/1742-6596/1194/1/012106.

[2] S. Mukai, *Simple Lie algebra and Legendre variety*, Nagoya Sūri Forum **3**, 1 (1996), https://www.math.nagoya-u.ac.jp/~mukai/paper/warwick15.pdf.

[3] P. Truini, *Exceptional Lie algebras, SU(3), and Jordan pairs*, Pacific J. Math. **260**, 227 (2012), doi:10.2140/pjm.2012.260.227.

[10] Correspondingly, $\mathcal{S}_q \equiv \mathcal{S}_q(P, \dot{P})\big|_{P=1,\dot{P}=0}$.

[4] A. Marrani and P. Truini, *Exceptional Lie algebras, SU(3) and Jordan pairs: Part 2. Zorn-type representations*, J. Phys. A: Math. Theor. **47**, 265202 (2014), doi:10.1088/1751-8113/47/26/265202.

[5] O. Loos, *Jordan pairs*, Springer, Berlin, Heidelberg, Germany, ISBN 9783540071662 (1975), doi:10.1007/BFb0080843.

[6] S. Ferrara, A. Marrani and B. Zumino, *Jordan pairs, E_6 and U-duality in five dimensions*, J. Phys. A: Math. Theor. **46**, 065402 (2013), doi:10.1088/1751-8113/46/6/065402.

[7] S. Ferrara, A. Marrani and M. Trigiante, *Super-Ehlers in any dimension*, J. High Energy Phys. **11**, 068 (2012), doi:10.1007/JHEP11(2012)068.

[8] J. M. Landsberg and L. Manivel, *Triality, exceptional Lie algebras and Deligne dimension formulas*, Adv. Math. **171**, 59 (2002), doi:10.1006/aima.2002.2071.

[9] K. McCrimmon, *Jordan algebras and their applications*, Bull. Amer. Math. Soc. **84**, 612 (1978), doi:10.1090/S0002-9904-1978-14503-0.

[10] P. Jordan, J. v. Neumann and E. Wigner, *On an algebraic generalization of the quantum mechanical formalism*, Ann. Math. **35**, 29 (1934), doi:10.2307/1968117.

[11] J. Tits, *Une classe d'algèbres de Lie en relation avec les algèbres de Jordan*, Indag. Math. **24**, 530 (1962), doi:10.1016/s1385-7258(62)50051-6.

[12] H. Freudenthal, *Sur le groupe exceptionnel E_7*, in *Hans Freudenthal: Selecta*, European Mathematical Society, Zurich, Switzerland, ISBN 9783037190586 (2009).

[13] M. Günaydin, K. Koepsell and H. Nicolai, *Conformal and quasiconformal realizations of exceptional Lie groups*, Commun. Math. Phys. **221**, 57 (2001), doi:10.1007/PL00005574.

[14] M. Gunaydin and O. Pavlyk, *Minimal unitary realizations of exceptional U-duality groups and their subgroups as quasiconformal groups*, J. High Energy Phys. **01**, 019 (2005), doi:10.1088/1126-6708/2005/01/019.

[15] M. Günaydin, *Lectures on spectrum generating symmetries and U-duality in supergravity, extremal black holes, quantum attractors and harmonic superspace*, in *The attractor mechanism*, Springer, Berlin, Heidelberg, Germany, ISBN 9783642107351 (2010), doi:10.1007/978-3-642-10736-8_2.

[16] A. N. Minchenko, *The semisimple subalgebras of exceptional Lie algebras*, Trans. Moscow Math. Soc. **67**, 225 (2006), doi:10.1090/S0077-1554-06-00156-7.

[17] N. Cantarini, A. Ricciardo and A. Santi, *Classification of simple linearly compact Kantor triple systems over the complex numbers*, J. Algebra **514**, 468 (2018), doi:10.1016/j.jalgebra.2018.08.009.

[18] P. Truini, M. Rios, A. Marrani, *The magic star of exceptional periodicity*, in *Nonassociative mathematics and its applications*, American Mathematical Society, Providence, USA, ISBN 9781470442453 (2019), doi:10.1090/conm/721.

[19] W. A. de Graaf, *Lie algebras: Theory and algorithms*, Elsevier, Amsterdam, Netherlands, ISBN 9780444501165 (2000), doi:10.1016/s0924-6509(00)x8001-4.

[20] V. G. Kac, *Infinite-dimensional Lie algebras*, Cambridge University Press, Cambridge, UK, ISBN 9780521372152 (1990), doi:10.1017/CBO9780511626234.

[21] R. D'Auria, S. Ferrara, M. A. Lledó and V. S. Varadarajan, *Spinor algebras*, J. Geom. Phys. **40**, 101 (2001), doi:10.1016/S0393-0440(01)00023-7.

[22] È. B. Vinberg, *The theory of convex homogeneous cones*, Trudy Moskov. Mat. Obšč. **12**, 303 (1963), http://mathscinet.ams.org/mathscinet-getitem?mr=158414.

[23] D. V. Alekseevsky and V. Cortés, *Special vinberg cones*, Transform. Groups **26**, 377 (2021), doi:10.1007/s00031-021-09649-w.

[24] D. V. Alekseevsky, A. Marrani and A. Spiro, *Special Vinberg cones and the entropy of BPS extremal black holes*, J. High Energy Phys. **11**, 100 (2021), doi:10.1007/JHEP11(2021)100.

[25] D. V. Alekseevsky, A. Marrani and A. Spiro, *Special Vinberg cones, invariant admissible cubics and special real manifolds*, (arXiv preprint) doi:10.48550/arXiv.2301.01168.

[26] C. H. Barton and A. Sudbery, *Magic squares and matrix models of Lie algebras*, Adv. Math. **180**, 596 (2003), doi:10.1016/S0001-8708(03)00015-X.

[27] B. de Wit, F. Vanderseypen and A. Van Proeyen, *Symmetry structure of special geometries*, Nucl. Phys. B **400**, 463 (1993), doi:10.1016/0550-3213(93)90413-J.

[28] S. Cecotti, *Homogeneous Kähler manifolds and T-algebras in $N = 2$ supergravity and superstrings*, Commun. Math. Phys. **124**, 23 (1989), doi:10.1007/BF01218467.

[29] J. C. Ferrar, *Strictly regular elements in Freudenthal triple systems*, Trans. Amer. Math. Soc. **174**, 313 (1972), doi:10.1090/S0002-9947-1972-0374223-1.