SciPost Phys. Proc. 14, 037 (2023)

Relativistic kinematics in flat and curved space-times

Patrick Moylan⋆

Department of Physics, The Pennsylvania State University,
Abington College, Abington, PA 19001 USA

⋆ pjm11@psu.edu

34th International Colloquium on Group Theoretical Methods in Physics
Strasbourg, 18-22 July 2022
doi:10.21468/SciPostPhysProc.14

Abstract

Almost immediately after the seminal papers of Poincaré (1905,1906) and Einstein (1905) on special relativity, wherein Poincaré established the full covariance of the Maxwell-Lorentz equations under the scale-extended Poincaré group and Einstein explained the Lorentz transformation using his assumption that the one-way speed of light in vacuo is constant and the same for all inertial observers (Einstein's second postulate), attempts were made to get at the Lorentz transformations from basic properties of space and time but avoiding Einstein's second postulate. Various such approaches usually involve general consequences of the relativity principle, such as a group structure to the set of all admissible inertial transformations and also assumptions about causality and/or homogeneity of space-time combined with isotropy of space. The first such attempt is usually attributed to von Ignatowsky in 1911. It was followed shortly thereafter by a paper of Frank and Rothe published in the same year. Since then, papers have continued to be written on the subject even up to the present. We elaborate on some of the results of such papers paying special attention to a 1968 paper of Bacri and Lévy-Leblond where possible kinematical groups include the de Sitter and anti-de Sitter groups and lead to special relativity in de Sitter and anti-de Sitter spaces.

Received 20-12-2022
Accepted 11-08-2023
Published 24-11-2023
doi:10.21468/SciPostPhysProc.14.037

1 Introduction

On Sept. 24, 1904, in a powerful and prophetic address to the *International Congress of Arts and Sciences* in St. Louis, Missouri, Henri Poincaré ushered in the *new relativity theory*.[1] Its

[1]The International Congress took place that year in St. Louis along with the other festivities of the 1904 World's Fair (Louisiana Purchase Exposition) celebrating the 100th anniversary of the Louisiana purchase of 1803.

foundation was the "*Principle of Relativity*," which, according to Poincaré, is [1]:

> *"That principle according to which the laws of physical phenomena should be the same, whether for an observer fixed, or for an observer carried along in a uniform movement of translation, so that we have not and could not have any means of discerning whether or not we are carried along in such a motion."*

Poincaré's lecture, appropriately delivered in the "New World," elevated the principle of relativity to a general law of physics on an equal footing with conservation of energy, which necessarily entails its universality. That this is such was boldly reasserted by him in Ref. [2], declaring that we shall "admit this law ... and admit [it] without restriction." The St. Louis lecture was published and widely read in academic circles worldwide in the ensuing months. A year later, in Albert Einstein's first paper on special relativity [3], there is found *without reference* a somewhat weaker and less precise rewording of Poincaré's statement of the relativity principle. In contrast to Poincaré, Einstein made no claim as to its universality.

Space-time is a four-dimensional Hausdorff manifold M with a smooth differentiable structure on it. Points in M correspond to events and curves to world lines of particles. Following Ehlers, Pirani and Schild (EPS) [4], [5] we take the curved analogs of straight world lines in affine space to be those curves (geodesics) which are "world lines of freely falling particles" and "behave *infinitesimally* like the straight lines of projective (or affine) four-space." A *symmetric affine connection* specifies the family of geodesics, with geodesics being curves whose "tangent directions" are "autoparallel." Translations are one-parameter groups of transformations with possibly only local C^∞ action on M, the geodesics being orbits of points under the action of the one-parameter translation subgroups.

With geodesics representing world lines of inertial observers and with inertial transformations being mappings between such geodesics, defined possibly only locally in some cases, our generalization of the relativity principle to curved space can be formulated in essentially the same way as Poincaré's statement of it. Just as in the affine case, the relativity principle demands that: (i) inertial transformations from one inertial frame to another take geodesics to geodesics and preserve parallelism of geodesics and (ii) "a group structure for the set of all inertial transformations" [6] at least in a local sense. We call the set of all such inertial transformations the relativity group or kinematical group of M. For the global formulation of Lie groups of transformations acting on a manifold, due in its local form to Sophus Lie, we refer the reader to [7].

2 Classification of possible kinematical groups

We assume that the kinematical or relativity group contains the rotation group $SO(3)$ as a subgroup. Furthermore, with translations defined as above and inertial boosts being defined as "uniform movements of translation," the kinematical group should be a subgroup of the Lie group of transformations formed out of *rotations*, *translations*, *inertial boosts* and *scale transformation* with (skew-symmetric) infinitesimal generators $\mathbf{L}_{ij}$, $\mathbf{P}_i$, $\mathbf{L}_{0i}$ $(i,j=1,2,3)$ and $\mathbf{S}$, respectively. Assume scale transformations commute with rotations and inertial boosts, and assuming rotational invariance, which implies that $\mathbf{P}_i$, $\mathbf{L}_{0i}$ are $SO(3)$ vector operators, we obtain

$$[\mathbf{L}_{ij},\mathbf{L}_{k\ell}]=-\delta_{ik}\mathbf{L}_{j\ell}-\delta_{i\ell}\mathbf{L}_{jk}+\delta_{jk}\mathbf{L}_{i\ell}+\delta_{j\ell}\mathbf{L}_{ik}\,, \tag{1}$$

$$[\mathbf{L}_{ij},\mathbf{L}_{0k}]=-\delta_{ik}\mathbf{L}_{0j}+\delta_{jk}\mathbf{L}_{0i}\,,\;[\mathbf{L}_{ij},\mathbf{P}_0]=0\,,\qquad[\mathbf{L}_{ij},\mathbf{P}_k]=-\delta_{ik}\mathbf{P}_j+\delta_{jk}\mathbf{P}_i\,, \tag{2}$$

$$[\mathbf{S},\mathbf{L}_{ij}]=0\,,\qquad[\mathbf{S},\mathbf{L}_{0i}]=0\,. \tag{3}$$

For the other brackets we have [8]:

$$[\mathbf{P}_0, \mathbf{P}_i] = \omega_i \mathbf{P}_0 + \gamma_{ij}\mathbf{P}_j + \frac{1}{2}\epsilon_{ik}\epsilon_{kmn}\mathbf{L}_{mn} + \alpha_{ik}\mathbf{L}_{0k} + \kappa_i \mathbf{S}\,, \tag{4}$$

$$[\mathbf{P}_i, \mathbf{P}_j] = \iota_{ij}\mathbf{P}_0 + \nu_{ijk}\mathbf{P}_k + \frac{1}{2}\mu_{ijk}\epsilon_{kmn}\mathbf{L}_{mn} + \psi_{ijk}\mathbf{L}_{0k} + \kappa_{ij}\mathbf{S}\,, \tag{5}$$

$$[\mathbf{L}_{0i}, \mathbf{P}_0] = \chi_i \mathbf{P}_0 + \lambda_{ij}\mathbf{P}_j + \frac{1}{2}\zeta_{ik}\epsilon_{kmn}\mathbf{L}_{mn} + \eta_{ik}\mathbf{L}_{0k} + \lambda_i \mathbf{S}\,, \tag{6}$$

$$[\mathbf{L}_{0i}, \mathbf{P}_j] = \rho_{ij}\mathbf{P}_0 + \pi_{ijk}\mathbf{P}_k + \frac{1}{2}\sigma_{ijk}\epsilon_{kmn}\mathbf{L}_{mn} + \tau_{ijk}\mathbf{L}_{0k} + \omega_{ij}\mathbf{S}\,, \tag{7}$$

$$[\mathbf{L}_{0i}, \mathbf{L}_{0j}] = \xi_{ij}\mathbf{P}_0 + \beta_{ijk}\mathbf{P}_k + \frac{1}{2}\lambda_{ijk}\epsilon_{kmn}\mathbf{L}_{mn} + \nu_{ijk}\mathbf{L}_{0k} + \tau_{ij}\mathbf{S}\,, \tag{8}$$

$$[\mathbf{S}, \mathbf{P}_0] = \alpha \mathbf{P}_0 + \beta_i \mathbf{P}_i + \frac{1}{2}\gamma_i \epsilon_{imn}\mathbf{L}_{mn} + \delta_i \mathbf{L}_{0i} + \zeta \mathbf{S}\,, \tag{9}$$

$$[\mathbf{S}, \mathbf{P}_i] = \alpha_i \mathbf{P}_0 + \beta_{ij}\mathbf{P}_j + \frac{1}{2}\gamma_{ik}\epsilon_{kmn}\mathbf{L}_{mn} + \delta_{ij}\mathbf{L}_{0j} + \eta_i \mathbf{S}\,. \tag{10}$$

The other brackets can be simplifed by further exploiting *rotational invariance* and *spatial isotropy* which implies that they must be expressible as linear combinations of the basic generators with the (rotationally) covariant tensor δ_{ij} and pseudo-tensor ϵ_{ijk} where δ_{ij} is the Kronecker delta and ϵ_{ijk} is the totally antisymmetric symbol. By using these facts we can rewrite Eqns. (4) to (10) as [8]

$$[\mathbf{P}_0, \mathbf{P}_i] = \omega_i \mathbf{P}_0 + \gamma \mathbf{P}_i + \frac{1}{2}\varepsilon\, \epsilon_{imn}\mathbf{L}_{mn} + \alpha \mathbf{L}_{0i} + \kappa_i \mathbf{S}\,, \tag{11}$$

$$[\mathbf{P}_i, \mathbf{P}_j] = \iota \delta_{ij}\mathbf{P}_0 + \nu \epsilon_{ijk}\mathbf{P}_k + \mu \mathbf{L}_{ij} + \psi \epsilon_{ijk}\mathbf{L}_{0k} + \kappa \delta_{ij}\mathbf{S}\,, \tag{12}$$

$$[\mathbf{L}_{0i}, \mathbf{P}_0] = \chi_i \mathbf{P}_0 + \lambda \mathbf{P}_i + \frac{1}{2}\zeta \epsilon_{imn}\mathbf{L}_{mn} + \eta \mathbf{L}_{0i} + \lambda_i \mathbf{S}\,, \tag{13}$$

$$[\mathbf{L}_{0i}, \mathbf{P}_j] = \rho \delta_{ij}\mathbf{P}_0 + \pi \epsilon_{ijk}\mathbf{P}_k + \sigma \mathbf{L}_{ij} + \tilde{\tau}\epsilon_{ijk}\mathbf{L}_{0k} + \omega \delta_{ij}\mathbf{S}\,, \tag{14}$$

$$[\mathbf{L}_{0i}, \mathbf{L}_{0j}] = \xi \delta_{ij}\mathbf{P}_0 + \beta \epsilon_{ijk}\mathbf{P}_k + \tilde{\lambda}\mathbf{L}_{ij} + \tilde{\nu}\epsilon_{ijk}\mathbf{L}_{0k} + \tau \delta_{ij}\mathbf{S}\,, \tag{15}$$

$$[\mathbf{S}, \mathbf{P}_0] = \alpha \mathbf{P}_0 + \beta_i \mathbf{P}_i + \frac{1}{2}\gamma_i \epsilon_{imn}\mathbf{L}_{mn} + \delta_i \mathbf{L}_{0i} + \zeta \mathbf{S}\,, \tag{16}$$

$$[\mathbf{S}, \mathbf{P}_i] = \alpha_i \mathbf{P}_0 + \tilde{\beta}\delta_{ij}\mathbf{P}_j + \frac{1}{2}\tilde{\gamma}\epsilon_{imn}\mathbf{L}_{mn} + \delta \mathbf{L}_{0i} + \eta_i \mathbf{S}\,. \tag{17}$$

Next consider the following automorphisms of the relativity group, G: the parity operator, Π, with action on $\mathfrak{g}$, the Lie algebra of G, given by

$$\Pi(\mathbf{P}_0) = \mathbf{P}_0\,, \quad \Pi(\mathbf{P}_i) = -\mathbf{P}_i\,, \quad \Pi(\mathbf{L}_{ij}) = \mathbf{L}_{ij}\,, \quad \Pi(\mathbf{L}_{0i}) = -\mathbf{L}_{0i}\,, \quad \Pi(\mathbf{S}) = \mathbf{S}\,, \tag{18}$$

and the time reversal operator, Θ, with action on $\mathfrak{g}$ given by

$$\Theta(\mathbf{P}_0) = -\mathbf{P}_0\,, \quad \Theta(\mathbf{P}_i) = \mathbf{P}_i\,, \quad \Theta(\mathbf{L}_{ij}) = \mathbf{L}_{ij}\,, \quad \Theta(\mathbf{L}_{0i}) = -\mathbf{L}_{0i}\,, \quad \Theta(\mathbf{S}) = \mathbf{S}\,. \tag{19}$$

Application of these automorphisms to the commutators, Eqns. (11) to (17) gives [8]

$$[\mathbf{P}_0, \mathbf{P}_i] = \alpha \mathbf{L}_{0i}\,, \tag{20}$$

$$[\mathbf{P}_i, \mathbf{P}_j] = \mu \mathbf{L}_{ij} + \kappa \delta_{ij}\mathbf{S} = \mu \mathbf{L}_{ij}\,, \tag{21}$$

$$[\mathbf{L}_{0i}, \mathbf{P}_0] = \lambda \mathbf{P}_i\,, \tag{22}$$

$$[\mathbf{L}_{0i}, \mathbf{P}_j] = \rho \delta_{ij}\mathbf{P}_0\,, \tag{23}$$

$$[\mathbf{L}_{0i}, \mathbf{L}_{0j}] = \tilde{\lambda}\mathbf{L}_{ij} + \tau \delta_{ij}\mathbf{S} = \tilde{\lambda}\mathbf{L}_{ij}\,, \tag{24}$$

$$[\mathbf{S}, \mathbf{P}_0] = \tilde{\alpha}\mathbf{P}_0\,, \tag{25}$$

$$[\mathbf{S}, \mathbf{P}_i] = \tilde{\beta}\mathbf{P}_i\,, \tag{26}$$

where we used the fact that the bracket is skew-symmetric to obtain $\kappa = \tau = 0$.

Proposition 1

$$\tilde{\alpha} = \tilde{\beta} = 1\,. \tag{27}$$

Proof: *Making use of commutators of Eqns. (20) to (26) we obtain*

$$\tilde{\beta}\mathbf{P}_i = [\mathbf{S},\mathbf{P}_i] = \frac{1}{\lambda}[\mathbf{S},[\mathbf{L}_{0i},\mathbf{P}_0]] = -\frac{1}{\lambda}[\mathbf{L}_{0i},[\mathbf{P}_0,\mathbf{S}]] = \frac{\tilde{\alpha}}{\lambda}[\mathbf{L}_{0i},\mathbf{P}_0] = \tilde{\alpha}\mathbf{P}_i\,,$$

which implies $\tilde{\alpha} = \tilde{\beta}$ *. To obtain* $\tilde{\alpha} = 1$*, use* $[\mathbf{L}_{01},\mathbf{P}_1] = \rho\mathbf{P}_0$ *to get*

$$\tilde{\alpha}\mathbf{P}_0 = [\mathbf{S},\mathbf{P}_0] = \frac{\tilde{\alpha}}{\rho}[\mathbf{S},[\mathbf{L}_{01},\mathbf{P}_1]] = \frac{\tilde{\alpha}}{\rho}[\mathbf{L}_{01},[\mathbf{S},\mathbf{P}_1]] = \frac{\tilde{\alpha}^2}{\rho}[\mathbf{L}_{01},\mathbf{P}_1] = \tilde{\alpha}^2\mathbf{P}_0\,.$$

Proposition 2 ***(****Bacry, Lévy-Leblond [8]****)***

$$\mu - \rho\alpha = 0\,, \tag{28}$$

$$\tilde{\lambda} - \rho\lambda = 0\,. \tag{29}$$

Proof: *Eq. (28) follows from the Jacobi identity*

$$[\mathbf{P}_i,[\mathbf{P}_j,\mathbf{L}_{0k}]] + [\mathbf{P}_j,[\mathbf{L}_{0k},\mathbf{P}_i]] + [\mathbf{L}_{0k},[\mathbf{P}_i,\mathbf{P}_j]] = 0\,,$$

together with $[\mathbf{L}_{ij},\mathbf{L}_{0k}] = -\delta_{ik}\mathbf{L}_{0j} + \delta_{jk}\mathbf{L}_{0i}$ *and the commutators before Proposition I. For Eq. (29) first use the the Jacobi identity*

$$[\mathbf{P}_0,[\mathbf{P}_i,\mathbf{L}_{0j}]] + [\mathbf{P}_i,[\mathbf{L}_{0j},\mathbf{P}_0]] + [\mathbf{L}_{0j},[\mathbf{P}_0,\mathbf{P}_i]] = 0\,,$$

and the commutators before Proposition I to obtain

$$\alpha\tilde{\lambda} - \lambda\mu = 0\,.$$

Then use this together with Eq. (28) to obtain Eq. (29).

One can show that the remaining Jacobi identities do not lead to any further independent constraints on the parameters in Eqns. (20) to (26) [8].

Propositions 1 and 2 imply the classification of admissible $\mathfrak{g}$ depends upon three independent real parameters ρ, α and λ. Let $\mathfrak{g} = \mathfrak{g}_{(\rho,\alpha,\lambda)}$. Then any admissible $\mathfrak{g}_{(\rho,\alpha,\lambda)}$ is isomorphic to $\mathfrak{g}_{(\rho,\alpha,\lambda)}$ with ρ, α, λ taking values 1 or 0. The explicit isomorphism is obtained by an appropriate scaling of generators, e.g. $\tilde{\mathbf{L}}_{0i} = \phi_\lambda(\mathbf{L}_{0i}) = \lambda^{-1/2}\mathbf{L}_{0i}$ with $\lambda > 0$ so that $[\tilde{\mathbf{L}}_{0i},\tilde{\mathbf{L}}_{0j}] = \dfrac{1}{\lambda}[\mathbf{L}_{0i},\mathbf{L}_{0j}] = \mathbf{L}_{ij}$. Thus, up to such isomorphisms, it suffices to restrict ρ, α, λ to values of 0 or 1. Following [8] we are led to the following cases:

Class R (relative time): $\rho = 1$:

$R1$. ($\alpha = 1$, $\lambda = 1$) From Eqns. (28) and (29) we have $\mu \neq 0$ and $\tilde{\lambda} \neq 0$ and from the commutation relations Eqns. (20) to (26) we obtain

$$\mathfrak{g}_{(1,1,1)} \cong \mathbb{C}\mathbf{S} \oplus_\tau \mathfrak{so}(5)\,,$$

with three possible real forms

$$\mathbb{R}\mathbf{S} \oplus_\tau \mathfrak{so}(5)\,, \qquad \mathbb{R}\mathbf{S} \oplus_\tau \mathfrak{so}(1,4)\,, \qquad \mathbb{R}\mathbf{S} \oplus_\tau \mathfrak{so}(2,3)\,,$$

where $\mathbb{C}\mathbf{S}$ and $\mathbb{R}\mathbf{S}$ respectively denote the one-dimensional Lie algebras over $\mathbb{C}$ and $\mathbb{R}$ generated by $\mathbf{S}$. ($\oplus_\tau$ means semidirect sum.)

*R*2. $(\alpha = 0,\ \lambda = 1 \Rightarrow \mu = 0,\ \tilde{\lambda} = 1)$

$$\mathfrak{g}_{(1,0,1)} \cong (\mathfrak{so}(4) \oplus \mathbb{C}\mathbf{S}) \oplus_\tau \mathfrak{t}^4\,,$$

where $\mathfrak{t}^4$ is the four dimensional abelian Lie algebra (ideal) over $\mathbb{C}$ generated by the $\mathbf{P}_0$, $\mathbf{P}_i$ $(i = 1,2,3)$. Up to isomorphism, all permissible real forms are[2]

$$(\mathfrak{so}(4) \oplus \mathbb{R}\mathbf{S}) \oplus_\tau \mathfrak{t}^4\,, \qquad \text{and} \qquad (\mathfrak{so}(1,3) \oplus \mathbb{R}\mathbf{S}) \oplus_\tau \mathfrak{t}^4\,,$$

where now $\mathfrak{so}(4)$ and $\mathfrak{t}^4$ are real Lie algebras. The case $(\mathfrak{so}(1,3) \oplus \mathbb{R}\mathbf{S}) \oplus_\tau \mathfrak{t}^4$ describes standard Lorentzian relativity exended by scale.

*R*3. $(\alpha = 1,\ \lambda = 0 \Rightarrow \mu = 1,\ \tilde{\lambda} = 0)$

$$\mathfrak{g}_{(1,1,0)} \cong \mathbb{C}\mathbf{S} \oplus_\tau \left\{ \widetilde{\mathfrak{so}}(4)_{(\mathbf{L}_{ij},\mathbf{P}_i)} \oplus_\tau \tilde{\mathfrak{t}}^4_{(\mathbf{L}_{0i},\mathbf{P}_0)} \right\},$$

where $\widetilde{\mathfrak{so}}(4)_{(\mathbf{L}_{ij},\mathbf{P}_i)} \oplus_\tau \tilde{\mathfrak{t}}^4$ is the semidirect sum of the $\mathfrak{so}(4)$ generated by $\mathbf{L}_{ij}$ and $\mathbf{P}_i$ $(i = 1,2,3)$ with an abelian Lie algebra $\tilde{\mathfrak{t}}^4_{(\mathbf{L}_{0i},\mathbf{P}_0)}$ over $\mathbb{C}$ generated by $\mathbf{P}_0$ and the $\mathbf{L}_{0i}$ $(i = 1,2,3)$. Permissible real forms (up to isomorphism) are

$$\mathbb{R}\mathbf{S} \oplus_\tau \left(\widetilde{\mathfrak{so}}(4)_{(\mathbf{L}_{ij},\mathbf{P}_i)} \oplus_\tau \tilde{\mathfrak{t}}^4_{(\mathbf{L}_{0i},\mathbf{P}_0)} \right), \qquad \text{and} \qquad \mathbb{R}\mathbf{S} \oplus_\tau \underbrace{\left(\widetilde{\mathfrak{so}}(1,3)_{(\mathbf{L}_{ij},\mathbf{P}_i)} \oplus_\tau \tilde{\mathfrak{t}}^4_{(\mathbf{L}_{0i},\mathbf{P}_0)} \right)}_{\textit{para-Poincaré Lie algebra}}.$$

*R*4. $(\alpha = 0,\ \lambda = 0 \Rightarrow \mu = 0,\ \tilde{\lambda} = 0)$ Lie algebra of the (scale-extended) Carroll Group: [8]

$$\mathfrak{g}_{(1,0,0)} \cong (\mathfrak{so}(3) \oplus_\tau \tilde{\mathfrak{t}}^3_{\mathbf{L}_{0i}} \oplus \mathbb{C}\mathbf{S}) \oplus_{\tilde{\tau}} \mathfrak{t}^4_{\mathbf{P}_\mu}\,.$$

There is only one acceptable real form. It is obtained by restricting $\mathfrak{g}_{(1,0,0)}$ to the reals. It is the (real) Lie algebra of the (scale-extended) Carroll group.

Class $\tilde{A}$ (absolute time): $\rho = 0$ $(\Rightarrow \mu = \tilde{\lambda} = 0)$:[3]

$\tilde{A}1$. $(\alpha = 0,\ \lambda = 1)$

$$\mathfrak{g}_{(0,0,1)} \cong \underbrace{\{\mathfrak{g}_{(\mathbf{L}_{ij},\mathbf{L}_{0k})} \oplus \mathbb{C}\mathbf{S}\}}_{\substack{\text{homogeneous Galilei Lie} \\ \text{algebra (scale extended)}}} \oplus_\tau \mathfrak{t}^4_{\mathbf{P}_0,\mathbf{P}_i}\,.$$

There is only one acceptable real form obtained by restricting $\mathfrak{g}_{(0,0,1)}$ to the reals. It is the (real) Lie algebra of the (scale-extended) inhomogeneous Galilei group.

$\tilde{A}2$. $(\alpha = 1,\ \lambda = 0)$

$$\mathfrak{g}_{(0,1,0)} \cong \{\{\mathfrak{so}(3)_{\mathbf{L}_{ij}} \oplus \mathbb{C}\mathbf{S}\} \oplus_\tau \mathfrak{t}^3_{\mathbf{P}_i}\} \oplus_\tau \mathfrak{t}^4_{(\mathbf{P}_0,\mathbf{L}_{0i})}\,.$$

It's easy to see that $\mathfrak{g}_{(0,1,0)} \cong \mathfrak{g}_{(0,0,1)}$ and that there is only one acceptable real form obtained by restricting $\mathfrak{g}_{(0,1,0)}$ to the reals.

$\tilde{A}3$. $(\alpha = 1,\ \lambda = 1)$

$$\mathfrak{g}_{(0,1,1)} \cong \mathbb{C}\mathbf{S} \oplus_\tau \mathfrak{n}\,.$$

$\mathfrak{n}$ is the ideal generated by $\mathbf{L}_{ij}$, $\mathbf{L}_{0i}$, $\mathbf{P}_0$, $\mathbf{P}_i$ and its two admissible real forms are the Lie algebras of the two Newton-Hooke groups [8].

$\tilde{A}4$. $(\alpha = 0,\ \lambda = 0)$

$$\mathfrak{g}_{(0,0,0)} \cong (\mathfrak{so}(3)_{\mathbf{L}_{ij}} \oplus \mathbb{C}\mathbf{S}) \oplus_\tau \mathfrak{t}^7_{(\mathbf{L}_{0i},\mathbf{P}_0,\mathbf{P}_i)}\,,$$

where $\mathfrak{t}^7_{(\mathbf{L}_{0i},\mathbf{P}_0,\mathbf{P}_i)}$ is the 7 dimensional abelian ideal generated by the $\mathbf{L}_{0i}$, $\mathbf{P}_0$, $\mathbf{P}_i$ $(i = 1,2,3)$. $\mathfrak{g}_{(0,0,0)}$ is the Lie algebra, extended by scale, of what is called the *static group* or *Aristotle group*.

[2]The reason why the real form containing $\mathfrak{so}(2,2)$ is not permitted is due to our assumption of rotational symmetry, which we made at the very start and which implies that admissible Lie algebras must contain the subalgebra $\mathfrak{so}(3)$.

[3]Our description of this class differs slighty from that in [8] so we put tildes on the *A*'s to distinguish them from Bacry and Lévy-Lebond's *A*'s in Ref. [8], i.e. $\tilde{A}1$ instead of *A*1 of Bacry, Lévy-Lebond etc.

3 Reduction of symmetry and non-compactness of the $e^{\nu \mathbf{L}_{0i}}$

The classification just given is at the Lie algebra level. The corresponding kinematical groups are obtained by suitable "exponentiation" [7] and restriction to subgroups [2]. As in [8], we consider only those cases for which any one-parameter subgroup of boosts "in any given direction forms a noncompact subgroup," i.e. the subgroups $e^{\nu \mathbf{L}_{0i}}$ are noncompact subgroups. This eliminates several of the listed real forms in the above classification. In particular, the first real form in the $R2$ case, which is $\{\mathfrak{so}(4)_{(\mathbf{L}_{ij},\mathbf{L}_{0i})} \oplus \mathbb{R}\mathbf{S}\} \oplus_\tau \mathfrak{t}^4$, is excluded as a possible kinematical group, since $\mathfrak{so}(4)_{(\mathbf{L}_{ij},\mathbf{L}_{0i})}$ is compact.

Following Poincaŕe in Ref. [2], we must, due to physical requirements, restrict the kinematical group to a subgroup. It should be a subgroup of the scale extended group with scale transformations depending upon the boost parameter, ν: "...we should consider only certain transformations in this group; we must assume that λ [the scale transformation] is a function of ν, and it is a question of choosing this function in such a way that this part of the group, which will be denoted by P, is itself a group [2]." For standard Lorentzian relativity (the second real form in the $R2$ case) this leads us to the result that $\lambda = \lambda_\nu = \pm 1$ (cf. [2]). Poincaré's argument for reduction of the scale extended Lorentz group to $SO_0(1,3) \times \Sigma_2 \cong SO(1,3)$ with $\Sigma_2 = \{\mathbb{I}_4, -\mathbb{I}_4\}$ runs as follows. Let

$$\Lambda(\nu) = \begin{pmatrix} \cosh\beta & \sinh\beta & 0 & 0 \\ \sinh\beta & \cosh\beta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

($\beta = \operatorname{arctanh}\nu$). "Any [homogeneous] transformation of the group P may be regarded as a transformation of the form $\lambda_\nu \Lambda(\nu)$ preceded and followed by suitable rotations" (KAK decomposition for scale extended $SO_0(1,3)$ restricted to the homogeneous part of P). We easily show that $R_\pi \lambda_\nu \Lambda(\nu) R_\pi^{-1} = \lambda_\nu \Lambda(-\nu)$ where R_π is a rotation about the y axis by π. Since the homogeneous part of P consists of all matrices of the form $\lambda_\nu \Lambda(R\nu)R'$ with $\nu \in \mathbb{R}$ and $R, R' \in SO(3)$, $\lambda_\nu \Lambda(-\nu)$ is in P. It will equal $\lambda_{-\nu}\Lambda(-\nu)$ for $\lambda_\nu = \lambda_{-\nu}$. So **$\boldsymbol{\lambda_\nu}$ should be an even function of $\boldsymbol{\nu}$.**

Now the inverse of $\lambda_\nu \Lambda(\nu)$ is $\lambda_\nu{}^{-1}\Lambda(-\nu)$. In order for this to be in P it must equal $\lambda_{-\nu}\Lambda(-\nu) = \lambda_\nu \Lambda(-\nu)$ which leads to $\lambda_\nu^{-1} = \lambda_\nu$. Hence $\lambda_\nu^2 = 1 \Rightarrow \lambda_\nu = \pm 1$ and, with $\rtimes$ denoting semidirect product, we have:

Theorem 1 *(Poincaré [2])*
Reduction of symmetry for the $SO_0(1,3)$ real form of Case R2 (scale extended $SO_0(1,3) \rtimes \mathbb{T}^4$) leads to $P = SO(1,3) \rtimes \mathbb{T}^4$, the proper inhomogeneous Lorentz group, as the kinematical group of special relativity. P contains space-time inversion $-\mathbb{I}_4$ and its connected component is $SO_0(1,3) \rtimes \mathbb{T}^4$.

Even though the homogeneous part of the Galilean group is not semisimple, Poincaré's arguments leading to Theorem 1 carry over to the scale extended Galilean group (Case $\tilde{A}1$) and they lead to the same conclusions, namely that $\{(SO(3) \rtimes N_3) \times \Sigma_2\} \rtimes \mathbb{T}^4$ is the kinematical group, where N_3 is the 3 dimensional subgroup of Galilean boosts and $\Sigma_2 = \{\mathbb{I}_4, -\mathbb{I}_4\}$, with $-\mathbb{I}_4$ being space-time inversion.

For the real forms $\mathbb{R}\mathbf{S} \oplus_\tau \mathfrak{so}(1,4)$ and $\mathbb{R}\mathbf{S} \oplus_\tau \mathfrak{so}(2,3)$ of Case $R1$, the situtation regarding reduction of scale is even more interesting. It is due to the fact that the connected components of the Lie groups associated with $\mathfrak{so}(1,4)$ and $\mathfrak{so}(2,3)$ have group decompositions into subgroups which involve $SO(1,3)$ instead of $SO_0(1,3)$ as one of the factors [9]. Since $SO(1,3)$ has two disconnected components, the generalization of Poincaré's argument to these cases is more complicated. It again leads to $\lambda_v = \pm 1$. However, we are free to set λ_v as $+1$ or -1 on either component. This leads to several choices for the relativity group, involving improper $O(1,4)$ or $O(2,3)$ transformations. Such additional structures could possibly lead to novel results in descriptions of elementary systems [10], [11] for relativistic quantum mechanics on de Sitter or anti-de Sitter space based on projective representations of $O(1,4)$ or $O(2,3)$, respectively. Although this is surely something well worth exploring, page limitations do not permit us to go further into the matter.

4 Other approaches and conclusion

There are other approaches to describing possible space-time structures and associated kinematical groups. The causality approach starts with a partial ordering on space time, M. Causal automorphisms on M are automorphisms which preserve the partial ordering. The set of all causal automorphisms forms a group, which, for M being an affine space, turns out to be the scale extended inhomogeneous orthochronous Lorentz group (Alexandrov-Zeeman result) [12], [13]. Lalan's 1937 classification [14] of all possible linear kinematics in two space-time dimensions compatible with the relativity principle is based on the Frank and Rothe paper [15]. Another very interesting approach going back to V. Gorini [16] rests on a physical assumption which essentially means that the set of inertial transformations taking frames at rest to frames at rest is the group $O(3)$. He proves that the only subgroups of $GL(4,\mathbb{R})$ satisfying this physical assumption are the proper orthochronous Galilean group and the proper orthochronous Lorentz group, along with isomorphic copies of it obtained by a rescaling of the boost generators [16].

In conclusion, incorporating scale symmetry into the analysis of classifications of possible kinematical groups leads to more interesting possible structures regarding discrete transformations like time reversal and spatial inversion, especially for the cases involving the de Sitter and anti-de Sitter groups.

References

[1] H. Poincaré, *L'état actuel et l'avenir de la physique mathématique*, Bull. Sci. Math. **28**, 302 (1904), https://gallica.bnf.fr/ark:/12148/bpt6k486258r/f476.item.

[2] H. Poincaré, *Sur la dynamique de l'électron*, Rend. Circ. Matem. Palermo **21**, 129 (1906), doi:10.1007/BF03013466.

[3] A. Einstein, *Zur Elektrodynamik bewegter Körper*, Ann. Phys. **322**, 891 (1905), doi:10.1002/andp.19053221004.

[4] J. Ehlers, F. A. E. Pirani, A. Schild, *General relativity; papers in honour of J. L. Synge*, Clarendon Press, Oxford, UK, ISBN 9780198511267 (1972).

[5] J. Ehlers and A. Schild, *Geometry in a manifold with projective structure*, Commun. Math. Phys. **32**, 119 (1973), doi:10.1007/BF01645651.

[6] J.-M. Lévy-Leblond, *One more derivation of the Lorentz transformation*, Am. J. Phys **44**, 271 (1976), doi:10.1119/1.10490.

[7] R. S. Palais, *A global formulation of the Lie theory of transformation groups*, Mem. Amer. Math. Soc. **22**, 10 (1957), doi:10.1090/memo/0022.

[8] H. Bacry and J.-M. Lévy-Leblond, *Possible kinematics*, J. Math. Phys. **9**, 1605 (1968), doi:10.1063/1.1664490.

[9] P. Moylan, *Parallelizations for induced representations of $SO_0(2,q)$*, in *Quantum theory and symmetries with lie theory and its applications in physics*, Springer, Singapore, ISBN 9789811327148 (2018), doi:10.1007/978-981-13-2715-5_12.

[10] E. P. Wigner, *On unitary representations of the inhomogeneous Lorentz group*, Ann. Math. **40**, 149 (1939), doi:10.2307/1968551.

[11] V. Bargmann and E. P. Wigner, *Group theoretical discussion of relativistic wave equations*, Proc. Natl. Acad. Sci. **34**, 211 (1948), doi:10.1073/pnas.34.5.211.

[12] A. D. Alexandrov, V. V. Ovchinnikova, *Notes on the foundations of relativity theory*, Vestnik Leningrad. Univ., **11** 95 (1953).

[13] E. C. Zeeman, *Causality implies the Lorentz group*, J. Math. Phys. **5**, 490 (1964), doi:10.1063/1.1704140.

[14] V. Lalan, *Sur les postulats qui sont à la base des cinemátiques* Bull. Soc. Math. France **65**, 83 (1937), http://eudml.org/doc/86692.

[15] P. Frank and H. Rothe, *Über die Transformation der Raumzeitkoordinaten von ruhenden auf bewegte Systeme*, Ann. Phys. **339**, 825 (1911), doi:10.1002/andp.19113390502.

[16] V. Gorini, *Linear kinematical groups*, Commun. Math. Phys. **21**, 150 (1971), doi:10.1007/BF01646749.