

The parastatistics of braided Majorana fermions

Francesco Toppan*

Centro Brasileiro de Pesquisas Físicas CBPF, Rua Dr. Xavier Sigaud 150,
Urca, cep 22290-180, Rio de Janeiro (RJ), Brazil.

* toppan@cbpf.br



34th International Colloquium on Group Theoretical Methods in Physics
Strasbourg, 18-22 July 2022

doi:[10.21468/SciPostPhysProc.14](https://doi.org/10.21468/SciPostPhysProc.14)

Abstract

This paper presents the parastatistics of braided Majorana fermions obtained in the framework of a graded Hopf algebra endowed with a braided tensor product. The braiding property is encoded in a t -dependent 4×4 braiding matrix B_t related to the Alexander-Conway polynomial. The nonvanishing complex parameter t defines the braided parastatistics. At $t = 1$ ordinary fermions are recovered. The values of t at roots of unity are organized into levels which specify the maximal number of braided Majorana fermions in a multiparticle sector. Generic values of t and the $t = -1$ root of unity mimic the behaviour of ordinary bosons.



Copyright F. Toppan.

This work is licensed under the Creative Commons

[Attribution 4.0 International License](https://creativecommons.org/licenses/by/4.0/).

Published by the SciPost Foundation.

Received 18-12-2022

Accepted 11-08-2023

Published 24-11-2023

doi:[10.21468/SciPostPhysProc.14.046](https://doi.org/10.21468/SciPostPhysProc.14.046)



Check for
updates

1 Introduction

Braided Majorana fermions have been intensively investigated since the [1] Kitaev's proposal that they can be used to encode the logical operations of a topological quantum computer which offers protection from decoherence (see also [2–4]). In this talk I present consequences and open questions about the parastatistics of \mathbb{Z}_2 -graded braided Majorana qubits derived from the results of [5]; this paper applied to \mathbb{Z}_2 -graded qubits the [6] framework of a graded Hopf algebra endowed with a braided tensor product. A nonvanishing complex braiding parameter t controls the spectra of multiparticle Majorana fermions. Inequivalent physics is derived for the set of t roots of unity which are organized into different levels ($L_2, L_3, \dots, L_\infty$). The levels interpolate between ordinary fermions (L_2 for $t = 1$) and the spectrum of bosons (" L_∞ " recovered at $t = -1$). The intermediate levels L_k for $k = 3, 4, 5, \dots$ implement a special type of parafermionic statistics (see [7–9]) which allows at most $k - 1$ braided Majorana excited states in any given multiparticle sector.

The paper is structured as follows. In Section 2 the braiding of \mathbb{Z}_2 -graded qubits is illustrated. In Section 3 the truncations of the spectra at roots of unity are discussed. The consequences for the parastatistics are presented in Section 4.

2 Braiding \mathbb{Z}_2 -graded qubits

We present the main ingredients of the construction. A single Majorana fermion can be described as a \mathbb{Z}_2 -graded qubit which defines a bosonic vacuum state $|0\rangle$ and a fermionic excited state $|1\rangle$:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (1)$$

The operators acting on the \mathbb{Z}_2 -graded qubit close the $\mathfrak{gl}(1|1)$ superalgebra. In a convenient presentation they can be defined as

$$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \delta = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \quad (2)$$

Their (anti)commutators are

$$\begin{aligned} [\alpha, \beta] &= \beta, & [\alpha, \gamma] &= -\gamma, & [\alpha, \delta] &= 0, & [\delta, \beta] &= -\beta, & [\delta, \gamma] &= \gamma, \\ \{\beta, \beta\} &= \{\gamma, \gamma\} = 0, & \{\beta, \gamma\} &= \alpha + \delta. \end{aligned} \quad (3)$$

The diagonal operators α, β are even, while β, γ are odd, with γ the fermionic creation operator.

The excited state is a Majorana since it is a fermion which coincides with its own antiparticle. This is a consequence of the fact that the (2) matrices span the Clifford algebra $Cl(2, 1)$ which, see [10, 11], is of real type (implying that the charge conjugation operator is the identity).

The construction of multiparticle \mathbb{Z}_2 -graded qubits is obtained via the coproduct Δ of the graded Hopf algebra $\mathcal{U}(\mathfrak{gl}(1|1))$, the Universal Enveloping Algebra of $\mathfrak{gl}(1|1)$.

The braiding of the graded qubits is realized by introducing a braided tensor product \otimes_{br} such that, for the operators a, b (\mathbb{I} is the identity) one can write

$$(\mathbb{I} \otimes_{br} a) \cdot (b \otimes_{br} \mathbb{I}) = \Psi(a, b), \quad (4)$$

where the right hand side operator $\Psi(a, b)$ satisfies braided compatibility conditions.

For the purpose of braiding \mathbb{Z}_2 -graded qubits it is only necessary to specify the braiding property of the creation operator γ :

$$(\mathbb{I} \otimes_{br} \gamma) \cdot (\gamma \otimes_{br} \mathbb{I}) = \Psi(\gamma, \gamma). \quad (5)$$

A consistent choice for the right hand side is to set

$$\Psi(\gamma, \gamma) = B_t \cdot (\gamma \otimes \gamma), \quad (6)$$

where B_t is a 4×4 constant matrix which depends on the complex parameter $t \neq 0$. The dot in the right hand side denotes the standard matrix multiplication.

The braiding compatibility condition is guaranteed by assuming B_t to be given by

$$B_t = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-t & t & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -t \end{pmatrix}, \quad (7)$$

since B_t satisfies

$$(B_t \otimes \mathbb{I}_2) \cdot (\mathbb{I}_2 \otimes B_t) \cdot (B_t \otimes \mathbb{I}_2) = (\mathbb{I}_2 \otimes B_t) \cdot (B_t \otimes \mathbb{I}_2) \cdot (\mathbb{I}_2 \otimes B_t). \quad (8)$$

The matrix B_t is the R -matrix of the Alexander-Conway polynomial in the linear crystal rep on exterior algebra [12] and is related, see [13], to the Burau representation of the braid group.

3 Truncations at roots of unity

The requirement that

$$B_t^n = \mathbb{I}_4, \tag{9}$$

for some $n = 2, 3, \dots$ finds solution for the $n - 1$ roots of the polynomial $b_n(t)$. This set of polynomials is defined as

$$b_{n+1}(t) = \sum_{j=0}^n (-t)^j,$$

so that

$$\begin{aligned} b_1(t) &= 1, \\ b_2(t) &= 1 - t, \\ b_3(t) &= 1 - t + t^2, \\ b_4(t) &= 1 - t + t^2 - t^3, \\ b_5(t) &= 1 - t + t^2 - t^3 + t^4, \\ &\dots = \dots \end{aligned}$$

The set of $b_k(t)$ polynomials enters the construction of multiparticle states. The n -particle vacuum $|0\rangle_n$ is given by the tensor product of n single-particle vacua:

$$|0\rangle_n = |0\rangle \otimes |0\rangle \otimes \dots \otimes |0\rangle \quad (n \text{ times}). \tag{10}$$

The fermionic excited states are created by applying powers of tensor products involving the single-particle creation operator γ . For $n = 2, 3$ one has, e.g., that the first excited state is created by

$$\begin{aligned} \gamma_{(2)} &= \mathbb{I}_2 \otimes_{br} \gamma + \gamma \otimes_{br} \mathbb{I}_2, \\ \gamma_{(3)} &= \mathbb{I}_2 \otimes_{br} \mathbb{I}_2 \otimes_{br} \gamma + \mathbb{I}_2 \otimes_{br} \gamma \otimes_{br} \mathbb{I}_2 + \gamma \otimes_{br} \mathbb{I}_2 \otimes_{br} \mathbb{I}_2. \end{aligned} \tag{11}$$

By taking into account the braided tensor product one obtains, for the second and third excited states,

$$\begin{aligned} \gamma_{(2)}^2 &= (1 - t) \cdot (\gamma \otimes_{br} \gamma), \\ \gamma_{(3)}^2 &= (1 - t) \cdot (\mathbb{I}_2 \otimes_{br} \gamma \otimes_{br} \gamma + \gamma \otimes_{br} \mathbb{I}_2 \otimes_{br} \gamma + \gamma \otimes_{br} \gamma \otimes_{br} \mathbb{I}_2), \\ \gamma_{(3)}^3 &= (1 - t)(1 - t + t^2) \cdot (\gamma \otimes_{br} \gamma \otimes_{br} \gamma). \end{aligned}$$

This construction works in general. The $b_k(t) = 0$ roots of the polynomials produce truncations at the higher order excited states and the corresponding spectrum of the theory.

4 The levels and the associated parastatistics

The single-particle Hamiltonian H can be identified with the operator δ in (2). It follows that the single-particle excited state has energy level $E = 1$. This is also true (due to the property of the Hopf algebra coproduct) for the first excited state in the multiparticle sector. Each creation operator produces a quantum of energy.

In the n -particle sector the energy spectrum of the theory depends on whether t produces a truncated or untruncated spectrum. The notion of truncation level acquires importance.

A “level- k ” root of unity, for $k = 2, 3, 4, \dots$, is a solution t_k of the $b_k(t_k) = 0$ equation such that, for any $k' < k$, $b_{k'}(t_k) \neq 0$.

The physical significance of a level- k root of unity is that the corresponding braided multiparticle Hilbert space can accommodate at most $k - 1$ Majorana spinors.

The special point $t = 1$, being the solution of the $b_2(t) \equiv 1 - t = 0$ equation, is a level-2 root of unity. It gives the ordinary total antisymmetrization of the fermionic wavefunctions. The $t = 1$ level-2 root of unity encodes the Pauli exclusion principle of ordinary fermions.

With an abuse of language, the $t = -1$ root of unity, which does not solve any $b_k(t) = 0$ equation, can be called a root of unity of ∞ level.

The physics does not depend on the specific value of t , but only on the root of unity level. A generic t which does not coincide with a root of unity produces the same untruncated spectrum of the $t = -1$ “ L_∞ ” level.

The following energy spectra are derived.

Case a, truncated L_k level: the n -particle energy eigenvalues E are

$$\begin{aligned} E &= 0, 1, \dots, n, & \text{for } n < k, \\ E &= 0, 1, \dots, k - 1, & \text{for } n \geq k; \end{aligned}$$

a plateau is reached for the maximal energy level $k - 1$; this is the maximal number of braided Majorana fermions that can be accommodated in a multiparticle Hilbert space;

Case b, untruncated ($t = -1$) L_∞ level: the n -particle energy eigenvalues E are

$$E = 0, 1, \dots, n, \quad \text{for any } n;$$

there is no plateau in this case. The energy eigenvalues grow linearly with N .

We can associate the roots of unity levels to fractions.

Let $t = e^{i\theta} = e^{if\pi}$ with $f \in [0, 2[$. The following fractions correspond to the roots of unity levels:

$$\begin{aligned} L_\infty &= 1, \\ L_2 &= 0, \\ L_3 &= \frac{1}{3}, \frac{5}{3}, \\ L_4 &= \frac{1}{2}, \frac{3}{2}, \\ L_5 &= \frac{1}{5}, \frac{3}{5}, \frac{7}{5}, \frac{9}{5}, \\ L_6 &= \frac{2}{3}, \frac{4}{3}, \\ L_7 &= \frac{1}{7}, \frac{3}{7}, \frac{5}{7}, \frac{9}{7}, \frac{11}{7}, \frac{13}{7}, \\ L_8 &= \frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \frac{7}{4}, \\ \dots &= \dots \end{aligned}$$

References

- [1] A. Y. Kitaev, *Fault-tolerant quantum computation by anyons*, Ann. Phys. **303**, 2 (2003), doi:[10.1016/S0003-4916\(02\)00018-0](https://doi.org/10.1016/S0003-4916(02)00018-0).
- [2] S. B. Bravyi and A. Y. Kitaev, *Fermionic quantum computation*, Ann. Phys. **298**, 210 (2002), doi:[10.1006/aphy.2002.6254](https://doi.org/10.1006/aphy.2002.6254).
- [3] C. Nayak, S. H. Simon, A. Stern, M. Freedman and S. Das Sarma, *Non-Abelian anyons and topological quantum computation*, Rev. Mod. Phys. **80**, 1083 (2008), doi:[10.1103/RevModPhys.80.1083](https://doi.org/10.1103/RevModPhys.80.1083).
- [4] L. H. Kauffman, *Knot logic and topological quantum computing with Majorana fermions*, in *Logic and algebraic structures in quantum computing*, Cambridge University Press, Cambridge, UK, ISBN 9781107033399 (2016), doi:[10.1017/CBO9781139519687](https://doi.org/10.1017/CBO9781139519687).
- [5] F. Toppan, *First quantization of braided Majorana fermions*, Nucl. Phys. B **980**, 115834 (2022), doi:[10.1016/j.nuclphysb.2022.115834](https://doi.org/10.1016/j.nuclphysb.2022.115834).
- [6] S. Majid, *Foundations of quantum group theory*, Cambridge University Press, Cambridge, UK, ISBN 9780511613104 (1995), doi:[10.1017/CBO9780511613104](https://doi.org/10.1017/CBO9780511613104).
- [7] G. Gentile, *Osservazioni sopra le statistiche intermedie*, Nuovo Cimento **17**, 493 (1940), doi:[10.1007/BF02960187](https://doi.org/10.1007/BF02960187).
- [8] H. S. Green, *A generalized method of field quantization*, Phys. Rev. **90**, 270 (1953), doi:[10.1103/PhysRev.90.270](https://doi.org/10.1103/PhysRev.90.270).
- [9] O. W. Greenberg and A. M. L. Messiah, *Selection rules for parafields and the absence of para particles in nature*, Phys. Rev. **138**, B1155 (1965), doi:[10.1103/PhysRev.138.B1155](https://doi.org/10.1103/PhysRev.138.B1155).
- [10] S. Okubo, *Real representations of finite Clifford algebras. I. Classification*, J. Math. Phys. **32**, 1657 (1991), doi:[10.1063/1.529277](https://doi.org/10.1063/1.529277).
- [11] H. L. Carrion, M. Rojas and F. Toppan, *Quaternionic and octonionic spinors. A classification*, J. High Energy Phys. **04**, 040 (2003), doi:[10.1088/1126-6708/2003/04/040](https://doi.org/10.1088/1126-6708/2003/04/040).
- [12] L. H. Kauffman and H. Saleur, *Free fermions and the Alexander-Conway polynomial*, Commun. Math. Phys. **141**, 293 (1991), doi:[10.1007/BF02101508](https://doi.org/10.1007/BF02101508).
- [13] N. Reshetikhin, C. Stroppel and B. Webster, *Schur-Weyl-type duality for quantized $gl(1|1)$, the burau representation of braid groups, and invariant of tangled graphs*, in *Perspective in analysis, geometry, and topology*, Birkhäuser, Boston, USA, ISBN 9780817682767 (2011), doi:[10.1007/978-0-8176-8277-4_16](https://doi.org/10.1007/978-0-8176-8277-4_16).
- [14] G. Lusztig, *Quantum groups at roots of 1*, Geom. Dedicata **35**, 89 (1990), doi:[10.1007/BF00147341](https://doi.org/10.1007/BF00147341).
- [15] C. de Concini and V. G. Kac, *Representations of quantum groups at roots of 1*, in *Operator algebras, unitary representations, enveloping algebras, and invariant theory*, Birkhäuser, Boston, USA, ISBN 9783764334895 (2000).
- [16] O. Viro, *Quantum relatives of the Alexander polynomial*, St. Petersburg Math. J. **18**, 391 (2007), doi:[10.1090/S1061-0022-07-00956-9](https://doi.org/10.1090/S1061-0022-07-00956-9).

- [17] A. Sartori, *The Alexander polynomial as quantum invariant of links*, Ark. Mat. **53**, 177 (2015), doi:[10.1007/s11512-014-0196-5](https://doi.org/10.1007/s11512-014-0196-5).
- [18] D. Gurevich and S. Majid, *Braided groups of Hopf algebras obtained by twisting*, Pacific J. Math. **162**, 27 (1994), doi:[10.2140/pjm.1994.162.27](https://doi.org/10.2140/pjm.1994.162.27).
- [19] D. Leites and V. Serganova, *Metasymmetry and Volichenko algebras*, Phys. Lett. B **252**, 91 (1990), doi:[10.1016/0370-2693\(90\)91086-Q](https://doi.org/10.1016/0370-2693(90)91086-Q).