Amplitude and colour evolution

Simon Plätzer

Institute of Physics, NAWI Graz, University of Graz, Universitätsplatz 5, A-8010 Graz, Austria
Particle Physics, Faculty of Physics, University of Vienna,
Boltzmanngasse 5, A-1090 Wien, Austria
Erwin Schrödinger Institute for Mathematics and Physics,
University of Vienna, A-1090 Wien, Austria

simon.plaetzer@uni-graz.at

Abstract

Colour evolution and parton branching at the amplitude level have become important theoretical frameworks to improve parton showers, and are algorithms in their own right: They complement shower development by resummation algorithms capable of including interference effects and subleading colour contributions at an unprecedented level. I summarize recent development in the field, focusing on soft gluon evolution, hadronization, and the CVolver framework.

1 Introduction

Event generators are central to phenomenology at colliders. Parton showers are the central component of event generators which describe the build-up of jets through energy loss of primary partons with large momenta of the order of some hard scale Q down to typical scales of $\mu_S \sim 1\text{GeV}$, where phenomenological models of hadronization describe how a partonic ensemble is forming the observed hadronic final state.

The accuracy of parton shower algorithms can only be assessed for certain (classes of) observables. Coherent branching algorithms such as the one underpinning the Herwig event generator [1] are known to predict global observables at next-to-leading logarithmic (NLL) accuracy, and dipole-type showers *e.g.* the Herwig implementations following [2,3], predict non-global observables at leading logarithmic (LL) accuracy. While all of these algorithms are based on the large-N_c limit (with $N_c = 3$ the number of colours in QCD), the structure of coherent branching is actually able to account for subleading colour provided that the structure of the hard process is sufficiently simple [4]. Dipole showers have recently been improved to reproduce properties of coherent branching and are thus now able to also predict global

observables at the NLL level, work which had been initiated in [5, 6]. With an additional algorithmic tweak [7,8] they can even reproduce the full-N_c colour factors as dictated by QCD coherence.

Several of the above mentioned refinements are based on approaches which analyse properties of amplitudes as they build up through successive radiation: Parton branching at the amplitude level [9, 10] originates from the study of multiple soft gluon emission [11, 12] and has become both a theoretical method as well as an algorithm in its own right which implements evolution at the amplitude level, or more precisely at the level of the cross section density operator which we shall introduce later. In this contribution, we will focus on how these algorithms can be constructed (at least in the soft gluon case), how we can use such constructions to assess the accuracy at which the evolution can predict certain observables, and how hadronization models could appear and be constrained within parton branching at the amplitude level.

2 Cross sections and cross section density operators

The aim of parton branching at the amplitude level is to calculate cross sections for a hard process accompanied by an additional number of partons which are either soft (with momentum components $\ll Q$), or collinear by building up the according scattering amplitude and squaring it at the end of the evolution. This is contrary to standard parton showers, which would iterate additional emissions in a probabilistic manner so long as they factorise at the level of the cross section for each individual additional radiation. Amplitude evolution is required whenever we cannot constrain the kinematics of additional radiation in a way that coherence arguments would allow to simplify the cross section. Colour correlations then unavoidably persist and can only be simplified further if one resorts to the large-N_c limit. The large-N_c limit itself is problematic since in a parametric counting for a logarithmic enhancement of $\alpha_S L^2 \sim 1$ (with L a large logarithm of the observable quantity), subleading logarithmic corrections might appear at the same level as subleading colour corrections since also $\alpha_s N_c^2 \sim 1$. Effects due to the exchange of gluons in the Glauber region can also not be accounted for in the large-N_c limit. Amplitude evolution algorithms rest on a decomposition of amplitudes in colour space,

$$|\mathcal{M}\rangle = \sum_\sigma \mathcal{M}_\sigma |\sigma\rangle, \tag{1}$$

where the abstract vectors $|\sigma\rangle$ (not to be confused with a true quantum mechanical state) span a space of colour structures, *i.e.* tensors of SU(N). In general, other quantum numbers as well as spin might be considered in a similar manner and we have recently been pointing out how such a formalism would be carried over to the entire Standard Model [13]. For n additional emissions on top of a hard process, amplitude evolution then considers the cross section density operator $\mathbf{A}_n$, which relates to the amplitude $|\mathcal{M}_n\rangle$ and the cross section with n additional partons as

$$\mathbf{A}_n = |\mathcal{M}_n\rangle\langle\mathcal{M}_n|, \qquad \mathrm{d}\sigma_n = \mathrm{Tr}[\mathbf{A}_n]\,\mathrm{d}\phi_n. \tag{2}$$

The trace is taken over colour structures, and the phase space integration $\mathrm{d}\phi_n$ is schematically only referring to the radiated partons in order to avoid notational clutter. The cross section density operator for producing $n + 1$ partons at a scale q can iteratively be built up as

$$\mathbf{A}_{n+1}(q) = \int_q^Q \frac{\mathrm{d}k}{k} \mathrm{P} e^{-\int_q^k \frac{\mathrm{d}k'}{k'} \Gamma_n(k')} \mathbf{D}_n(k) \mathbf{A}_n(k) \mathbf{D}_n^\dagger(k) \overline{\mathrm{P}} e^{-\int_q^k \frac{\mathrm{d}k'}{k'} \Gamma_n^\dagger(k')}. \tag{3}$$

Several choices of evolution variables are possible. $\mathbf{D}_n$ describes the emission of the $n + 1$st gluon (subject to the ordering variable, *e.g.* energy) for which the colour structures 'grow'
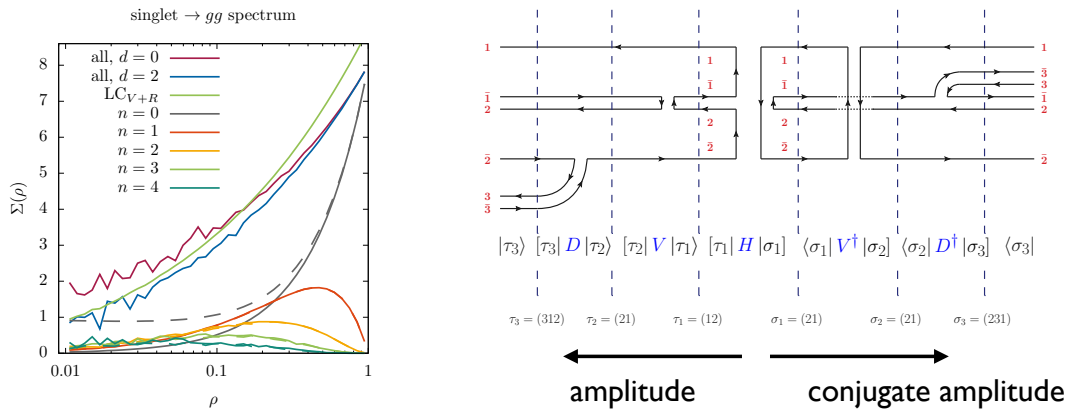
Figure 1: Example results from amplitude evolution in the colour flow basis. We calculate the gaps-between-jets cross section with a veto scale ρ for cone-like jets. The evolution proceeds by a Monte Carlo over colour flows resulting from gluon exchange and emission in the amplitude and conjugate amplitude, allowing all histories to interfere with each other. Here V denotes the insertion of the virtual evolution operator resulting from the exponentiation of $\mathbf{\Gamma}$ and H is the initial condition provided by the amplitude of the hard scattering process. Figures amended from [15].

into a larger colour space, and $\mathbf{\Gamma}_n$ encodes one-loop virtual exchanges, which mix different colour structures into each other [11, 14]. These exchanges are taken into account to all orders by means of the ordered exponential. In practice we resort to the colour flow basis in colour space, which allows for a convenient organization of an expansion around the large-N_c limit [11, 12], including in principle arbitrary higher orders in $1/N_c$. The process of gluon emission and exchange in the colour flow basis is now known to two emissions, and two loops, respectively [14]. The CVolver library [11,15] has been developed to solve amplitude evolution equations in a new kind of parton shower algorithm, and we have recently provided first results for jet vetoes in Z and Higgs decays, with further applications in progress. We illustrate some results and the basic principles in Fig. 1. The work on amplitude evolution has also brought up the question to what extent the mixing of colour structures through soft gluon exchanges can be related to colour reconnection models. In fact, in [16], we have demonstrated that this might be the case and the kinematic dependence, and the mechanism of Baryon production through colour reconnection introduced in [17] have a one-to-one correspondence with the picture of soft gluon evolution. How we could include hadronization effects in the amplitude evolution paradigm has up until now remained elusive, and will be addressed in the remainder of this contribution.

3 Constructing amplitude evolution

The cross section and evolution equations we have been advocating above are relevant to jet cross sections where we can democratically sum over colour configurations. This also implies that infrared divergences, for infrared safe observables, will always cancel subject to the cyclicity of the trace (see [4] how the cyclicity and multi-parton colour correlations can be used to extract subleading-N_c radiation patterns). We will now generalize this picture by addressing exclusive final states, which is the target of event generators and detailed predictions at the hadron level. Since now projections of partonic systems onto certain hadronic systems come

into play, the most general from of the cross section should be considered to be

$$d\sigma_m = \sum_n \int \alpha_0^n \text{Tr}[\mathbf{M}_n \mathbf{U}_{nm}] d\phi_n d\phi_m. \tag{4}$$

In this case $\mathbf{U}_{nm}$ will be the operator which projects onto the observed hadronic systems (labelled by m) and contains the observable function, and $\mathbf{M}_n$ is the cross section density operator before any infrared and ultraviolet divergences have been removed. In this expression, we integrate over possible partonic momentum configurations $d\phi_n$ and sum over partonic multiplicities n. In case of a jet cross section, say, $\mathbf{U}_{nm}$ would be proportional to the unit operator in colour space and would relate partonic to hadronic final states directly assuming that hadronization would be a small correction. In the analysis below we shall work with the effective measurement on parton level, thus introducing

$$\mathbf{U}_n = \sum_m \int \mathbf{U}_{nm} d\phi_m \qquad \text{and} \qquad \sigma = \sum_m \int d\sigma_m = \sum_n \int \alpha_0^n \text{Tr}[\mathbf{M}_n \mathbf{U}_n] d\phi_n. \tag{5}$$

In general $\mathbf{U}_n$ defines an infrared unsafe cross section, as implied by the hadronic final state we want to probe to formulate how hadronization can enter the amplitude level evolution paradigm. In [18] we have detailed how a redefinition of the measurement operator $\mathbf{U}_n$ in terms of a finite, though possibly non-perturbative, measurement operator $\mathbf{S}_n$, along with the standard redefinition of the bare coupling α_0 in terms of the renormalized $\alpha_S(\mu_R)$ provides both UV and IR subtractions for $\mathbf{M}_n$. A further redefinition of the density operator $\mathbf{M}_n$ in terms of a then finite density operator $\mathbf{A}_n$ provides an expression of the cross section in terms of resummed and finite quantities. The cross section is in particular renormalization group invariant both with respect to the UV renormalization scales μ_R and the resolution scales $\vec{\mu}_s$ which separate out unresolved radiation. In an event generator, there would be one scale μ_S which is the parton shower infrared cutoff at which hadronization takes over from parton shower evolution, but this is a certain choice and needs to be looked at more general. If the redefinitions have not been truncated at fixed order in $\alpha_S(\mu_R)$, the resulting cross section is

$$\sigma = \sum_n \int \alpha_S^n(\mu_R) \text{Tr}[\mathbf{A}_n(\mu_R, \vec{\mu}_S) \mathbf{S}_n(\mu_R, \vec{\mu}_S)] d\phi_n, \tag{6}$$

and thus is independent of any of the resolution scales. A residual resolution scale dependence arises at fixed order, and can be related to a tower of subleading logarithms in the evolution [18]. Evolution equations in all of the separation scales are implied. Within our formalism, we have the advantage that we can relate these infrared resolution criteria to the classes of observables in question and thus judge how accurate our algorithm will be able to generate resummed predictions. More importantly, the RGE evolution of $\mathbf{A}_n$,

$$\partial_S \mathbf{A}_n = \mathbf{\Gamma}_{n,S} \mathbf{A}_n + \mathbf{A}_n \mathbf{\Gamma}_{n,S}^\dagger - \sum_{s \geq 1} \alpha_S^s \mathbf{R}_{S,n}^{(s)} \mathbf{A}_{n-s} \mathbf{R}_{S,n}^{(s)\dagger}, \tag{7}$$

exactly resembles the parton branching at amplitude level algorithm we have been introducing in the first part, where $\mathbf{R}_n^{(s)}$ now describes emissions of s partons on top of n already radiated partons and does itself include virtual corrections, e.g. a one-loop correction to emission of a single gluon. The subscript S on the derivatives simply indicates the relation to the specific component of $\vec{\mu}_S$ we have been differentiating to. The evolution equation for a non-trivial $\mathbf{S}_n$ (with a similar notation as in Eq. 7, $[dp_i]\tilde{\delta}(p_i)$ indicating the phase space of an emitted parton i),

$$\partial_S \mathbf{S}_n = -\tilde{\mathbf{\Gamma}}_{S,n}^\dagger \mathbf{S}_n - \mathbf{S}_n \tilde{\mathbf{\Gamma}}_{S,n} + \sum_{s \geq 1} \alpha_S^s \int \tilde{\mathbf{R}}_{S,n+s}^{(s)\dagger} \mathbf{S}_{n+s} \tilde{\mathbf{R}}_{S,n+s}^{(s)} \prod_{i=n+1}^{n+s} [dp_i]\tilde{\delta}(p_i), \tag{8}$$

can be seen as an evolution equation of a hadronization model: the hard density matrix $\mathbf{A}_n$ evolves from a large scale Q down to the infrared scales, and from lower to higher multiplicity, while $\mathbf{S}_n$ evolves from small scales, and thus in principle a non-perturbative initial condition, and from larger to smaller multiplicities. Both objects also do evolve in opposite directions in colour space dimensionality. In the evolution of $\mathbf{S}_n$, we recover in particular the mechanism of colour reconnection mentioned in the first part. Other features depend on what model is used to express the non-perturbative initial condition needed for $\mathbf{S}_n$. If, in particular, the a cluster-model like *Ansatz* is used, then additional gluons can be absorbed in the evolution of $\mathbf{S}_n$ in the form of soft $q\bar{q}$ pairs, thus resembling the cluster fission mechanism [19]. The picture which emerges puts the parton shower infrared cutoff (if there is a single resolution scale) onto the same conceptual level as the factorization scale for parton distribution functions, and indicates that the hard parton shower evolution, and the evolution of the hadronization model shall be matched at this scale. Any residual dependence on $\vec{\mu}_S$ would then serve as a measure of uncertainty. In fact, by perturbatively expanding out the evolution equation of $\mathbf{S}_n$, one can quantify how accurately a given observable, subject to some infrared resolution, will be predicted by the algorithm, see [18] for more details. In a first step towards precision [18] also calculates all relevant structure up to the second order and all anomalous dimensions and emission operators are known to the two-loop level in the soft case [14], and we have made first steps to extent this beyond the soft limit to systematically include collinear contributions, as well [20].

4   Conclusion

In this contribution we have presented an overview of parton shower development centred around the amplitude level evolution paradigm. This framework is used both as a theoretical tool as well as an algorithm in its own right, which is able to calculate observables in a subleading-N_c accurate way. We have focused on how such an approach can be derived from a renormalization group point of view, which gives rise to identifying how a hadronization model can be included and constrained from amplitude evolution. Colour reconnection models based on such an approach are directly reproduced, as are features of cluster fission. We plan to further use this insight to extent the hadronization models in Herwig, and to supplement the CVolver approach by hadronization corrections.

Acknowledgements

I am grateful to Jeff Forshaw, Jack Holguin, Maximilian Löschner, Ines Ruffa and Malin Sjödahl for numerous discussions on this subject and the fruitful collaboration on many related aspects.

Funding information   This work was in part supported by the European Union's Horizon 2020 research and innovation programme as part of the Marie Sklodowska-Curie Innovative Training Network MCnetITN3 (grant agreement no. 722104). I also thank the Erwin Schrödinger Institute at Vienna for support through numerous Research-in-Teams programs RIT2020, RIT0421 and RIT0521.

References

[1] S. Gieseke, P. Stephens and B. Webber, *New formalism for QCD parton showers*, J. High Energy Phys. **12**, 045 (2003), doi:10.1088/1126-6708/2003/12/045.

[2] S. Plätzer and S. Gieseke, *Coherent parton showers with local recoils*, J. High Energy Phys. **01**, 024 (2011), doi:10.1007/JHEP01(2011)024.

[3] S. Plätzer and S. Gieseke, *Dipole showers and automated NLO matching in Herwig++*, Eur. Phys. J. C **72**, 2187 (2012), doi:10.1140/epjc/s10052-012-2187-7.

[4] J. R. Forshaw, J. Holguin and S. Plätzer, *Rings and strings: A basis for understanding subleading colour and QCD coherence beyond the two-jet limit*, J. High Energy Phys. **05**, 190 (2022), doi:10.1007/JHEP05(2022)190.

[5] M. Dasgupta, F. A. Dreyer, K. Hamilton, P. F. Monni, G. P. Salam and G. Soyez, *Parton showers beyond leading logarithmic accuracy*, Phys. Rev. Lett. **125**, 052002 (2020), doi:10.1103/PhysRevLett.125.052002.

[6] J. R. Forshaw, J. Holguin and S. Plätzer, *Building a consistent parton shower*, J. High Energy Phys. **09**, 014 (2020), doi:10.1007/JHEP09(2020)014.

[7] K. Hamilton, R. Medves, G. P. Salam, L. Scyboz and G. Soyez, *Colour and logarithmic accuracy in final-state parton showers*, J. High Energy Phys. **03**, 041 (2021), doi:10.1007/JHEP03(2021)041.

[8] J. Holguin, J. R. Forshaw and S. Plätzer, *Improvements on dipole shower colour*, Eur. Phys. J. C **81**, 364 (2021), doi:10.1140/epjc/s10052-021-09145-1.

[9] Z. Nagy and D. E. Soper, *A parton shower based on factorization of the quantum density matrix*, J. High Energy Phys. **06**, 097 (2014), doi:10.1007/JHEP06(2014)097.

[10] J. R. Forshaw, J. Holguin and S. Plätzer, *Parton branching at amplitude level*, J. High Energy Phys. **08**, 145 (2019), doi:10.1007/JHEP08(2019)145.

[11] S. Plätzer, *Summing large-N towers in colour flow evolution*, Eur. Phys. J. C **74**, 2907 (2014), doi:10.1140/epjc/s10052-014-2907-2.

[12] R. A. Martínez, M. De Angelis, J. R. Forshaw, S. Plätzer and M. H. Seymour, *Soft gluon evolution and non-global logarithms*, J. High Energy Phys. **05**, 044 (2018), doi:10.1007/JHEP05(2018)044.

[13] S. Plätzer and M. Sjodahl, *Amplitude factorization in the electroweak Standard Model*, (arXiv preprint) doi:10.48550/arXiv.2204.03258.

[14] S. Plätzer and I. Ruffa, *Towards colour flow evolution at two loops*, J. High Energy Phys. **06**, 007 (2021), doi:10.1007/JHEP06(2021)007.

[15] M. De Angelis, J. R. Forshaw and S. Plätzer, *Resummation and simulation of soft gluon effects beyond leading color*, Phys. Rev. Lett. **126**, 112001 (2021), doi:10.1103/PhysRevLett.126.112001.

[16] S. Gieseke, P. Kirchgaeßer, S. Plätzer and A. Siodmok, *Colour reconnection from soft gluon evolution*, J. High Energy Phys. **11**, 149 (2018), doi:10.1007/JHEP11(2018)149.

[17] S. Gieseke, P. Kirchgaeßer and S. Plätzer, *Baryon production from cluster hadronisation*, Eur. Phys. J. C **78**, 99 (2018), doi:10.1140/epjc/s10052-018-5585-7.

[18] S. Plätzer, *Colour evolution and infrared physics*, J. High Energy Phys. **07**, 126 (2023), doi:10.1007/JHEP07(2023)126.

[19] M. Bähr et al., *Herwig++ physics and manual*, Eur. Phys. J. C **58**, 639 (2008), doi:10.1140/epjc/s10052-008-0798-9.

[20] M. Löschner, S. Plätzer and E. S. Dore, *Multi-emission kernels for parton branching algorithms*, (arXiv preprint) doi:10.48550/arXiv.2112.14454.