In this note, we present an example where the presence of the conserved currents of spin $3/2$ does not guarantee supersymmetry.

To this end, we apply the Jordan-Wigner transformation to a tensor product of two $c = 1$ RCFTs, $\mathcal{B} = (U(1)_4/\mathbb{Z}_2)^2$. We show that the corresponding fermionic theory $\mathcal{F}$ has a conserved current of spin $3/2$ but non-constant Ramond-Ramond partition function. Furthermore, the SUSY unitarity bound is violated in $\mathcal{F}$.

Let us start with a brief review on $U(1)_4/\mathbb{Z}_2$: It is a $\mathbb{Z}_2$ orbifold of a compact boson on $S_1$ with radius 2 where $\mathbb{Z}_2$ acts on the scalar field $\varphi(z)$ as $i : \varphi(z) \rightarrow -\varphi(z)$. The $\mathbb{Z}_2$ orbifold theory consists of nine chiral primaries summarized in Table 1. Here, the character of each primary can be expressed in terms of $\chi_{k,n}$ and $\theta_{\alpha,\beta}$:

$$\chi_{k,n} = \frac{1}{\eta(q)} \sum_{m \in \mathbb{Z}} q^{(k+2mn)^2/4n}, \quad \theta_{\alpha,\beta} = \frac{1}{\eta(q)} \sum_{m \in \mathbb{Z}} q^{(m+\alpha)^2} e^{2\pi im\beta}. \quad (1)$$

Next, we consider a tensor product theory $\mathcal{B} = (U(1)_4/\mathbb{Z}_2)^2$. We denote 81 primaries of this theory as $\varphi_{i,j} \equiv \varphi_i \otimes \varphi_j$ for $i,j = 0, 2, \cdots, 8$ and their characters as $\chi_{\varphi_{i,j}}$. The bosonic theory $\mathcal{B}$ has a non-anomalous $\mathbb{Z}_2$ symmetry generated by the Verlinde line defect associated with the primary $\varphi_{2,1} \equiv \varphi_2 \otimes \varphi_1$ of conformal weight $h = 3/2$. Performing the Jordan-Wigner transformation with the above non-anomalous $\mathbb{Z}_2$ symmetry, one can compute the partition functions of $\mathcal{F}$. As a result, we show that the NS-sector partition function involves of 21 primaries and the NS vacuum character is given by $f_{0}^{\text{NS}} = \chi_{\varphi_{0,0}} + \chi_{\varphi_{2,1}}$. This implies that $\mathcal{F}$ has spin $3/2$ currents.

On the other hand, the Ramond-Ramond partition function of $\mathcal{F}$ becomes

$$Z_{F}^{R}(\tau, \bar{\tau}) = |\chi_{\varphi_{0,5}} - \chi_{\varphi_{2,7}}|^2 + |\chi_{\varphi_{0,6}} - \chi_{\varphi_{2,8}}|^2 + |\chi_{\varphi_{0,6}} - \chi_{\varphi_{1,6}}|^2 + |\chi_{\varphi_{3,5}} - \chi_{\varphi_{1,7}}|^2 + |\chi_{\varphi_{0,8}} - \chi_{\varphi_{1,8}}|^2 + |\chi_{\varphi_{3,6}} - \chi_{\varphi_{1,8}}|^2 + |\chi_{\varphi_{4,0}} - \chi_{\varphi_{1,4}}|^2 + |\chi_{\varphi_{3,5}} - \chi_{\varphi_{7,7}}|^2 + |\chi_{\varphi_{5,5}} - \chi_{\varphi_{7,8}}|^2, \quad (2)$$

<table>
<thead>
<tr>
<th>Label</th>
<th>Character</th>
<th>$h$</th>
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</thead>
<tbody>
<tr>
<td>$\varphi_0$</td>
<td>$\frac{1}{2} \chi_{0,2} + \frac{1}{4} \theta_{0,\frac{1}{2}}$</td>
<td>0</td>
<td>$\varphi_3$</td>
<td>$\frac{1}{2} \chi_{2,2}$</td>
<td>$\frac{1}{2}$</td>
<td>$\varphi_6$</td>
<td>$\frac{1}{2} \theta_{1,0} + \frac{1}{2} \theta_{1,\frac{3}{2}}$</td>
<td>$\frac{1}{16}$</td>
</tr>
<tr>
<td>$\varphi_1$</td>
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<td>1</td>
<td>$\varphi_4$</td>
<td>$\chi_{1,2}$</td>
<td>$\frac{1}{8}$</td>
<td>$\varphi_7$</td>
<td>$\frac{1}{2} \theta_{1,0} - \frac{1}{2} \theta_{1,\frac{3}{2}}$</td>
<td>$\frac{9}{16}$</td>
</tr>
<tr>
<td>$\varphi_2$</td>
<td>$\frac{1}{2} \chi_{2,2}$</td>
<td>$\frac{1}{2}$</td>
<td>$\varphi_5$</td>
<td>$\frac{1}{2} \theta_{1,0} + \frac{1}{2} \theta_{1,\frac{3}{2}}$</td>
<td>$\frac{1}{16}$</td>
<td>$\varphi_8$</td>
<td>$\frac{1}{2} \theta_{1,0} - \frac{1}{2} \theta_{1,\frac{3}{2}}$</td>
<td>$\frac{9}{16}$</td>
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</table>

Table 1: Primaries of an orbifold theory $U(1)_4/\mathbb{Z}_2$. 
where the $q$-expansion of each character is given as follows,

\[
\begin{align*}
f_0^R(q) &= \chi_{\bar{\varphi}_{0,5}} - \chi_{\varphi_{2,7}} = \chi_{\varphi_{0,6}} - \chi_{\varphi_{2,8}} = \chi_{\bar{\varphi}_{0,6}} - \chi_{\varphi_{1,6}} \\
&= q^{-1/48} + q^{95/48} + q^{143/48} + 2q^{191/48} + 2q^{239/48} + 3q^{287/48} + \cdots , \\
f_1^R(q) &= \chi_{\varphi_{3,5}} - \chi_{\bar{\varphi}_{1,7}} = \chi_{\varphi_{0,8}} - \chi_{\varphi_{1,8}} - \chi_{\varphi_{3,6}} - \chi_{\varphi_{1,8}} \\
&= q^{23/48} + q^{71/48} + q^{119/48} + q^{167/48} + 2q^{215/48} + 2q^{263/48} + \cdots , \\
f_2^R(q) &= \chi_{\varphi_{0,4}} - \chi_{\varphi_{1,4}} = \chi_{\varphi_{5,5}} - \chi_{\varphi_{7,7}} = \chi_{\bar{\varphi}_{5,6}} - \chi_{\bar{\varphi}_{7,8}} \\
&= q^{1/24} + q^{25/24} + q^{49/24} + 2q^{73/24} + 2q^{97/24} + 3q^{121/24} + \cdots .
\end{align*}
\]

We note that the three functions $f_0^R(q)$, $f_1^R(q)$, and $f_2^R(q)$ are identical to the characters of the Ising model. Therefore, $\frac{1}{2}Z_F^R(\tau, \bar{\tau})$ is same with the partition function of the Ising model. In other words, the contribution from the excited states to $Z_F^R$ is not cancelled out, the fermionic theory $\mathcal{F}$ could not be supersymmetric. Furthermore, we can see that the supersymmetry unitary bound is violated in $\mathcal{F}$. 

2