The study of partially accessible Markov networks is of major interest in stochastic thermodynamics. Gaining new insights into statistics of observables and estimators for entropy production is not only a timely challenge but also crucial from an operational point of view.

The present work adapts a transition-based description for partially accessible Markov networks to apply known results from martingale theory and extant entropy estimators to currents counting transitions along a single edge in a Markov network. The idea is not novel, but generalizes previous results that are based on martingale theory (e.g. Ref. [22]).

In a numerical case study for a molecular motor model, the corresponding entropy estimator is compared to known estimators like the thermodynamic uncertainty relation, the first passage ratio and a naive bound based on the Kullback-Leibler divergence. The deduced entropy estimator is interpreted as an effective affinity, a concept already introduced in Ref. [30].

The manuscript is well written, both in terms of grammar and mathematical rigor. It provides an introduction and a summary of the main results that give the reader a clear overview over the paper and how it relates to recent literature. While the theoretical sections of the paper are kept at a quite technical level, the paper gives explicit demonstrations of the main results in a numerical case study.

Our Reply: We thank the Referee for carefully reading the manuscript and providing comments to improve the manuscript.

Major concerns:

It is unclear to me how the paper concludes that the derived estimator "can be significantly more accurate than those based on ... Kullback-Leibler divergences".

Equation (112) is the only one in the manuscript that is claimed to be an estimator based on the Kullback-Leibler divergence. However, it is certainly not the only estimator of this sort, since more sophisticated versions have been proved e.g. in Ref. [49] and particularly in Ref. [48], which discusses an entropy estimator for the identical molecular motor model..

In light of these recent advances, it is, in my opinion, not justified to claim greater accuracy without a comparison to these recent bounds. The authors should either clarify these misleading conclusions or include these more recent entropy estimators in the numerical study. (see also the minor remark below)

Our Reply: We would like to stress that we do not claim greater accuracy than bounds based on the Kullbac-Leibler divergence. What we aimed to say is that the modified infimum ratio is more accurate than the naive estimator $\hat{s}_{\rm KL}$ of the Kullback-Leibler divergence Eq. (1). In retrospect, we agree with the Referee that the last sentence of the abstract is not clear enough. Therefore we have modified it into:

Moreover, we show that estimates of dissipation based on extreme value statistics can be significantly more accurate than those based on thermodynamic uncertainty ratios, as well as, those based on a naive estimator obtained by neglecting nonMarkovian correlations in the Kullback-Leibler divergence of the trajectories of the integrated edge current.

Note that all the information of the arrow of time given by J is contained in the Kullback-Leibeler divergence

$$\left\langle \ln \frac{\mathrm{d}\mathbb{P}_{\rm ss}[J_0^t]}{\mathrm{d}(\mathbb{P}_{\rm ss}\circ\Theta)[J_0^t]} \right\rangle_{\rm ss}. \tag{1}$$

Hence, the more accurate this quantity is estimated, the better of a bound that is obtained. The estimator $\hat{s}_{\rm KL}$ is obtained from Eq.(1) by neglecting nonMarkovian statistics in the currents J. Reference [48,49] indeed consider more sophisticated estimates of (1) that involve waiting time statistics, and thus nonMarkovian statistics in J.

To further clarify this issue, we have added a paragraph at the end of Section 8.3 of the new manuscript:

"*As discussed, none of the estimators considered in this paper capture dissipation of the process near the stalling state. However, it should be emphasized that the estimator $\hat{s}_{\rm KL}$ is a crude approximation of the Kullback-Leibler* *divergence Eq. (116), as it ignores nonMarkovian correlations in the trajectories of J. As shown in Refs. [49-51], by considering nonMarkovian effects, such as the statistics of the transition times along the edge, a fraction of the dissipation can be estimated, even at the stalling state.* "

Minor remarks:

2. Two novel entropy estimators, the infimum ratio and the modified infimum ratio, are introduced. From an operational point of view, it is interesting to know how much statistics is actually needed for these estimators, i.e. how long should the trajectory be for comparable error/bias? Is there a connection to the results obtained in [18]?

Our Reply: Indeed, this has been discussed in the paper "*Izaak Neri, Estimating entropy production with first-passage times, 2022 J. Phys. A: Math. Theor. 55 304005*".

The estimation of p_- requires $n_s \sim 1/p_- = \exp(a^*\ell)$ samples, and it becomes thus increasingly difficult to estimate $p_-(\ell)$ at larger values of ℓ. However this problem is resolved with the modified infimum ratio, as $\hat{\hat{s}}_{\rm inf}$ lower bounds $\dot{s}$ at finite values of ℓ.

We discuss this now more carefully in Discussion:

"*The estimator $\hat{s}_{\rm inf}(\ell)$ lower bounds the rate of dissipation in the limit of large ℓ. However, in this limit*

$$p_-(\ell) = \exp\left(-a^*\ell[(1+o(\ell)]\right), \tag{2}$$

where the prefactor $a^* > 0$ is the effective affinity, and therefore the number of samples $n_s \sim 1/p_-$ required to estimate p_- increases exponentially in ℓ, see Ref. [18], and we have called this the infinite threshold problem. However, in this Paper we have shown that for edge currents the average rate of dissipation can be estimated from the extreme value statistics of a current at finite thresholds ℓ through the estimator $\hat{\hat{s}}_{\rm inf}$ (see Eq. (9)). This resolves, for the case of edge currents, the problem with infinite thresholds with estimators $\hat{s}_{\rm inf}(\ell)$ based on extreme values, which is a special case of the first passage problem considered in Refs. [10,15,18]. "

3. The math in Ch. 5 seems to be very similar to the derivations leading to the "informed partial" estimator in Gili Bisker et al J. Stat. Mech. (2017) 093210. Both incorporate the effective affinity of Ref [30] and the stalling distribution (cf. the paragraph following eq. (63)). How deep is this connection? More concrete, it might be more appropriate to numerically compare to the entropy estimators discussed in Bisker et al. These estimators seem more closely related to the present work than those that are based on waiting times.

Our Reply: We thank the Referee for mentioning Gil Bisker et al. J. Stat. Mech. (2017) 093210, and hence giving us the opportunity to clariy the relation between our work and this paper.

Indeed, this paper is relevant, and we are afraid to have omitted it in the previous version of the manuscript. Therefore, we would like to thank the Referee for pointing this important point out.

Reference Gil Bisker et al. J. Stat. Mech. (2017) studies two estimators of dissipation, the so-called average *passive partial entropy production* rate and the so-called average *informed partial entropy production* rate. The former equals $\hat{s}_{\rm kl}$ and the latter $\hat{\hat{s}}_{\rm inf} = \lim_{\ell\to\infty}\hat{s}_{\rm inf}(\ell) = a^*_{x\to y}\overline{j}_{x\to y}$ in the present manuscript.

To point this out, we have added the following paragraph to the Introduction:

"*The quantity $\overline{j}a^*$ has also been studied in Ref. [33], where it is called the average informed partial entropy production rate. Reference [33] shows that $\overline{j}a^*$ is a better estimate of dissipation than a naive estimator $\hat{s}_{\rm KL}$ obtained from neglecting nonMarkovian correlations in the Kullback-Leibler divergence of the trajectories of the current, and which is called the average passive partial entropy production rate, i.e., $\dot{s} \geq \overline{j}a^* \geq \hat{s}_{\rm KL}$. However, Reference [33] argues that $\overline{j}a^*$ cannot be measured passively by observing the trajectory of a current, and instead should be determined as the force at which the edge current stalls, which can be measured actively if we can exert a microscopic force on the system. In the present paper, we show that the informed partial entropy production rate, $\overline{j}a^*$, can be measured passively through the modified infimum ratio $\hat{\hat{s}}_{\rm inf}(\ell)$, as $\hat{\hat{s}}_{\rm inf}(\ell) = \overline{j}a^*$ for $\ell \in \mathbb{N}$.* "

Reference Gil Bisker et al. J. Stat. Mech. (2017) shows that $a^*_{xy}\bar{j}_{x\to y} \geq \hat{s}_{\mathrm{KL}}$, however Gil Bisker et al. J. Stat. Mech. (2017) do not provide a means to measure $a^*_{x\to y}$ from the passive observation of the trajectory of $J^{x\to y}$. Indeed, Gil Bisker et al. J. Stat. Mech. (2017) states:

"*Despite this similarity, the extra content embodied in the informed partial entropy production allows one to capture more of the underlying dissipation. The main challenge of this approach, however, is that the stalling force may be difficult to access in an experimental setup: isolating precise control of the transition rates only over the observed link may not be possible, as one might expect, for example, when monitoring a complex chemical reaction network within a living cell. When it is application, however, the informed partial entropy production offers a better estimate of the total entropy production rendering it a more useful inference tool; especially, for unicyclic networks where it captures all of the entropy production*"

In other words, $a^*_{x\to y}$ can be determined by finding the stalling force at which the edge current $\bar{j}_{x\to y}$ stalls, but this requires the observer to be able to exert such a force.

On the other hand, with the modified infimum ratio we can measure $a^*_{x\to y}$, and thus the informed partial entropy production, through a passive measurement of the trajectories of $J_{x\to y}$.