1 Answer

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Although announcing (in the abstract and introduction) a study
of clocks in relativity, this manuscript actually proposes
short analysis of the standard massive spinning particle.
Besides vague speculations in the introduction and conclusion,
the goal appears to be to study the Lagrangian and Hamiltonian
formulation of a particle carrying a fundamental
representation of the Poincaré group, i.e. fixed mass and spin.
This is standard mechanics, it is hardly original and the
presentation by the authors show a misconception of the problem.

Indeed, what they call a singular Lagrangian is more
simply symptomatic of a gauge symmetry, coming from first class
constraints, as known and expected. If one were to follow the
logic developed by the atuhors (more precisely, section 3),
one would discard the standard (geodesic) Lagrangian for the
(massive spinless) relativistic particle. In fact, it is known
-standard text book physics- how to deal with such a Lagrangian.
The Hamiltonian is a linear combination of the constraints,
the coefficients in front of those constraints are Lagrange
multipliers and can be chosen arbitrarily (with, of course,
appropriate smoothness and monotonicity assumptions).

Such a choice here of u1,u2,u3 and u4 of eqn (3)) amounts to a
gauge fixing of the constraints, this yields Lagrangian whose
equations of motion are well-defined and describe the evolution
of the gauge-fixed system prescribed by the choice of Lagrange
multipliers. Due to the weakness and non-originality of the
analysis, I can not recommend this manuscript for publication.
If the authors persevere in this line of research, I would urge
them to revise their work in light of the well-known Hamiltonian
formulation of constrained systems, and gauge symmetries,
and to solve the evolution equations explicitly to actually
discuss clock properties.
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The logic presented in the paper in no way discards the standard Lagrangian for a relativistic massive point particle. If we use the logic presented in Section 3 for a relativistic massive (spinless) point particle and remove all spurious degrees of freedom from the Lagrangian that describes it, so that only physical degrees of freedom will be present in the Lagrangian, it will be regular. In the case of the KLS-Staruszkiewicz Lagrangian, after leaving only physical degrees of freedom characteristic of a genuine rotator with 5 degrees of freedom, it will remain singular.

The rest of the paper describes the singularity of the inverse Legendre transformation (not the singularity of the Lagrangian of the KLS-Staruszkiewicz particle) and the consequent additional constraints on velocity for a system that has both Hamiltonian constraints built in from the start. Hence we get a new Lagrangian (to our best knowledge, not present in literature) that is the counterpart to the Lagrangian of a structureless point particle with intrinsic motion at the speed of light (while the counterpart of the KLS-Staruszkiewicz particle is the ordinary Lagrangian of a massive particle with subluminal intrinsic motion).

We explain our point in more detail below. First we discuss what is known in the context of our paper then we discuss new things.

We are aware of Dirac's Generalized Hamiltonian Dynamics and the role of first and second-class constraints in defining the Hamiltonian and obtaining the resulting equations of motion. We are also aware of the method of passing from the Hamiltonian to the Lagrangian formulation of dynamics (also described by Dirac).

What the Referee is describing as true for the spinning particle model and known from standard textbooks is just our starting point: first we identify four constraints characteristic of the simplest possible spinning particle model when described in terms of spatio-temporal position x^μ and a single null direction. This can be summarized as follows: there are 4 first-class constraints: A: $k{\cdot}k = 0$ (k is null), B: $k \cdot \pi = 0$ (projection invariance – independence of the scale of k, where π is the momentum canonically conjugate with k) /this means that there are only 2 spherical degrees of freedom used to describe the spinning degree of freedom of a relativistic rotator/, C: $p \cdot p = m^2$ (the usual mass constraint with p being four-momentum canonically conjugate with x) and D: $w \cdot w = -\frac{1}{4}m^4 l^2$ with w being the Pauli-Lubanski spin pseudo-vector (then $w \cdot w$ is proportional to Gram determinant of vectors k, p, π) /m is the parameter of mass and l is the parameter of length (or spin $s = \frac{1}{2}ml$)/. All of these 4 constraints are first class, hence the total Hamiltonian, as implied by Diracs' method is simply

$$H = c_1\left(p \cdot p - m^2\right) + c_2\left((\pi \cdot \pi)(k \cdot \pi)^2 + \frac{1}{4}m^2 l^2\right) + c_3 k \cdot \pi + c_4 k \cdot k,$$

which is a linear combination of functionally independent 4 first-class constraints /equivalent to A, B, C, D/ with c_1, c_2, c_3, c_4 being arbitrary functions (non-physical gauge degrees of freedom). This Hamiltonian form for the simplest possible spinning particle model has been arrived at in the literature (modulo notation) in at least three independent ways [1, 2, 3]. The way of derivation of the above Hamiltonian form, as described above, seems to be the simplest and most direct. Counting the number of degrees of freedom (as indicated by Dirac) gives 4 for the number of physical degrees of freedom, which is 1 less than expected for a classical genuine rotator. And this is our starting point – in spacetime, the particle is represented by a cylinder of radius l, that is, not by a single wordline but by a family of arbitrary worldlines confined to this cylinder. The physical state is uniquely defined by momenta (leading to a well-behaved quantum system upon quantization, as described in [1]), however the classical trajectory, as implied by Hamiltonian equations written in terms of coordinates x, y, z, θ, ϕ characteristic of a classical rotator, remains indeterminate (as described in [4]).

We can say, that this system with 4 degrees of freedom has its physical state uniquely determined in the 'momentum space' – when we interpret the physical state as one which is determined by the direction of the timelike p and the direction of the spacelike w/m (Pauli-Lubański) vector, fixed m and $\frac{1}{2}ml$, respectively, for the two vectors, and by their mutual orthogonality. And this interpretation seems to be coherent with the success of deriving quantum equations for a spinning system as presented by KLS model [1].

Historically, the KLS paper [1] comes first. More than a decade later, in a quite different context the KLS Lagrangian is independently rediscovered by Staruszkiewicz [5] – in the context of **ideal clock**, and the KLS model again draws attention (at the Lagrangian level it was not a priori obvious that the worldline and the clocking frequency is indeterminate). Then several works appeared.

Namely, people started to ask, whether, the system could be interpreted as a classical genuine rotator with a unique worldline (e.g. it is known that the uniqueness of the worldline could be regained /though not necessarily/ by introducing interaction with external fields, as then – again according to Diracs' method – this would introduce a new consistency condition of the interaction form with the already existing constraints, that would make the coordinates x, y, z, θ, ϕ unique as functions of the time t - this idea was partly realized in [6]– however the standard coupling with the electromagnetic field used therein seems questionable and not fully solving the problem [4, 7]). As already said, there are 4 physical degrees of freedom and 4 gauge degrees of freedom (in fact three, because there is arbitrariness in choosing the arbitrary parametrization along the worldline's path – the arbitrariness characteristic of any relativistic dynamical system, let's call it 'trivial' gauge freedom). One can work in a gauge in which $c_4 = 0$ and $c_3 = 0$ from the start and assume $k.k = 0$ and $k.\pi = 0$ as additional auxiliary conditions preserved by the Hamiltonian. One can then assume a 3rd gauge-fixing function $f(u_1, u_2) = 0$ such that the independent parametrization (that is, the 'trivial' gauge) is effectively chosen to be the proper time in the rest frame of p (the time of the center of mass frame) or another choice, say, that the parametrization is the coordinate time of Minkowski's inertial coordinates (in which the momentum p has nonzero spatial components). Let denote the parameter by t. The functions $x(t), y(t), z(t), \theta(t)$ and $\phi(t)$ remain not independent of each other, contrary to what one would expect for a genuine rotator (of course, this result is obvious at the Hamiltonian level, because the considered system with constrained mass and constrained spin has 4 physical degrees of freedom, not 5 as for a genuine rotator – the indeterminacy of motion at the level of Lagrangian can be noticed by examining the Hessian determinant in a general family of rotators [4].

This all above is known and this is what the Referee seems to point to and indicates as lack of novelty and, therefore, deficiency of our present work. However, all this is not what is the main content and result of our present work. We can rewrite our manuscript to make

this point more clear for the reader.

Our point is the following. Is there a physically meaningful way to fix the unwanted gauge degree of freedom that makes the worldline and clocking rate indeterminate and then reinterpret the system with 4 physical degrees of freedom as a genuine rotator with determinate helical worldline and fixed intrinsic frequency?

Our answer is yes, and this leads to a new Lagrangian form explicitly written, which has not been considered in the context of spinning particles. We agree with the Referee that setting $c_1 = c_2$ may look as it were a mere gauge fixing, but we think it is something more. Note, that the only Lagrangian form explicitly written in the literature and possibly to be written in the class of models defined by the above Hamiltonian, is the KLS Lagrangian considered in [1]. It was assumed as the starting point from which one can identify constraints and write down the corresponding general Hamiltonian. In our present manuscript we ask the question about passing in the reverse direction: what forms of functionally independent Lagrangians one down when one starts with the general Hamiltonian Starting from a Lagrangian, identifying momentum constraints and writing down the Hamiltonian can be called the direct Legendre transform. Starting form a Hamiltonian, identifying velocity constraints and writing down the Lagrangian we call the inverse Legendre transform.

Now, starting with the general Hamiltonian as written above (with four quite arbitrary gauge functions c_1, c_2, c_3, c_4) we investigate the analytical properties of the inverse Legendre transform (with no assumptions made about velocities). Then one can see, that the corresponding Jacobian of the map between Lorentz scalars in generalized momenta space (characteristic of the Hamiltonian) and Lorentz scalars in generalized velocities space, and corresponding to this inverse Legendre transform, has its rank dependent on the gauge functions c_1, c_2, c_3, c_4, however only c_1, c_2 have a meaning for the uniqueness of the worldline as a path in spacetime (as such, independent of the still arbitrary parametrization – 'trivial' gauge). Depending on the rank of the map, one obtains 1) a Lagrangian known in the Literature which is analytically suitable for considering subluminal motions /for which the kinematical part of momentum $\frac{\dot{x}}{\sqrt{\dot{x}\cdot\dot{x}}}$ has a meaning/ and 2) a Lagrangian which is analitically suitable to considering motion with the velocity of light for which $\dot{x} \cdot \dot{x} = 0$. Note, that these Lagrangians are analytically distinct and their role cannot be interchanged. It is also important that the structure of cones in Minkowski spacetime must have its analytical consequence at the Lagrangian level – in the velocity space – (velocities on the null cone and velocities on the Lobachevsky unit hyperboloid belong to geometrically distinct objects).

This analyticall difference resembles the one for structureless point relativistic particle. In this case we have a general Hamiltonian $H = \frac{c_1}{2}(p \cdot p - m^2)$ as a starting point. We have $L = p\cdot\dot{x} - H = \frac{1}{2}\left(\dot{x}\cdot\dot{x}/c_1 + c_1 m^2\right)$ is the corresfroming Lagrangian. The Lagrangian equation $\partial_{c_1} L = 0$ gives the equivalent reduced Lagrangian $L = m\sqrt{\dot{x}\cdot\dot{x}}$ only if $m \neq 0$ and this Lagrangian gives analytically well behaved momentum only if $\dot{x}\cdot\dot{x} \neq 0$. If $m = 0$ then the Lagrangian reads

$L = \frac{1}{2c_1}\dot{x}\cdot\dot{x}$ with the primary velocity constraint $\dot{x}\cdot\dot{x} = 0$ that follows from the Lagrangian equation $\partial_{c_1} L = 0$ and with c_1 remaining an arbitrary degree of freedom during motion (the one corresponding to the reparametrization invariance).

References

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