Kinetic Theory of *Tilted* Dirac Cone Materials

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We formulate the Boltzmann kinetic equations for interacting electrons with tilted Dirac cones in two space dimensions characterized by a tilt parameter $0 \leq \zeta < 1$. By solving the linearized Boltzmann equation, we find that the broadening of the Drude pole is enhanced by $\kappa(\zeta) \times (1 - \zeta^2)^{-1/2}$, where the κ is interaction-induced enhancement factor. The intensity of the Drude pole is also anisotropically enhanced by $(1 - \zeta^2)^{-1}$. The ubiquitous "redshift" factors $(1 - \zeta^2)^{1/2}$ can be regarded as a manifestation of an underlying spacetime structure in such solids. The additional broadening κ that arises from the interactions can not be obtained from a simple coordinate change and are more pronounced for electrons in a ζ-deformed Minkowski spacetime of tilted Dirac fermions.

I. INTRODUCTION

Boltzmann equations are powerful method to study the transport and optical properties of electrons/hole plasmas [1]. Within this method one can address the coupling of charge carriers to impurities, phonons and their interactions with themselves [2, 3]. Within this method, the relativisitc fermions can also be addressed [4], enabling the study of transport in Dirac materials examplified in two dimensions by graphene [5]. Furthermore, Berry phase effects can be easily accomodate within this method [6]. Application of kinetic theory to Dirac/Weyl materials in three space that host chiral fermions[1] can powerfully capture the chiral anomaly and its consequences [7, 8].

When the chemical potential in a Dirac material is zero, and it is defectless [9], it serves as a realization of a quantum critical system [10, 11], where having set the chemical potential $\mu = 0$, the physics is entirely determined by the ration of $k_B T$ and $\hbar\omega$, the frequency at which the system is being probed. In the undoped Dirac materials, the kinetic theory can be used to study the ration η/s of the viscosity and entropy density that suggests graphene as a perfect fluid [12, 13], which has been indeed evidenced in experiments [14–16].

A variant of Dirac materials are materials with tilted Dirac cone in their spectrum that can occur in various classes of materials from 8Pmmn-borohpene [17, 18] to MoS_2 family [19]. Original observation of tilted Dirac cone dispersion was in the organig compound (BEDT-TTF)$_2$I$_3$ [20–22]. The Hamiltonian of these systems (in 2+1 dimensional spacetime) is given by

$$H = v_F \boldsymbol{\sigma}.\boldsymbol{p} + v_F \boldsymbol{\zeta}.\boldsymbol{p} \tag{1}$$

where the second term is characterized by a vector-looking dimensionless parameter $\boldsymbol{\zeta}$ that quantifies the tilting of the Dirac cone and we limit our focus to $|\boldsymbol{\zeta}| < 1$ situation. The above innocent looking energy dispersion has far reaching consequnces. To see how, let us start with the Hamiltonian

of an upright Dirac cone given by $H' = v_F \boldsymbol{\sigma}.\boldsymbol{p}$. Now imagine the "coordinate transformations" $t' = t$, $\boldsymbol{x}' = \boldsymbol{x} + v_F \boldsymbol{\zeta} t$ known as Gallilean transformation, which implies $\partial_{\boldsymbol{x}'} = \partial_{\boldsymbol{x}}$ and $\partial_{t'} = \partial_t + v_F \boldsymbol{\zeta}.\boldsymbol{\partial}_x$. The last equation readily suggests $H' = H + v_F \boldsymbol{\zeta}.\boldsymbol{p}$ which is nothing but Eq. (1). Affecting the above coordinate transformation on the Minkiwski metric $\eta_{ab} = \text{diag}(-1, 1, 1)$ or $ds^2 = -v_F^2 dt'^2 + (d\boldsymbol{x}')^2$ of the Dirac materials gives,

$$ds^2 = -v_F^2 dt^2 + (d\boldsymbol{x} - v_F \boldsymbol{\zeta} dt)^2. \tag{2}$$

This is indeed the emergent metric $g_{\mu\nu}$ that consistently describes the tilted Dirac/Weyl cone materials [23–27]. The matrix representation of the metric (2) can be diagonalized by an orthogonal matrix e_μ^a as

$$g_{\mu\nu} = e_\mu^a e_\nu^b \eta_{ab}. \tag{3}$$

The above equation in fact shows that the Dirac fermions in tilted Dirac cone materials are equpped with "frame fields" e_μ^a [28–30]. This indicates that the tilt is in fact a proxy for the frame fields, hence a deep gravitational analogy is hidden in tilted Dirac cone materials.

Therefore it is timely to formulate a kinetic theory for the fermions in tilted Dirac materials [31]. As in earlier works [11, 12] we consider the undoped case $\mu = 0$, where like any other quantum critical system the relaxation time of electron-electron interaction (Coulomb interaction) is controlled by the $k_B T$ and the frequency ω at which the system is probed as $\tau_{ee}^{-1} \sim \alpha^2 \frac{k_B T}{\hbar}$ [10, 13]. On the other hand, the relaxation time for a (dilute) density of charged impurities is given by [32] $\tau_{imp}^{-1} \sim \frac{1}{\hbar} \frac{(\frac{Ze^2}{\epsilon_r})^2 \rho_{imp}}{\max[k_B T, \mu]}$, where Ze is the charge of an random impurity and ρ_{imp} is the average spatial density of impurities. We see that in the high temperature limit $k_B T \gg \mu$ (which specifies the appropriate regime for Dirac fluid and applies to our case) and for a dilute density of impurities, the Coulomb interaction dominates the relaxation mechanism ($\tau_{ee}^{-1} \gg \tau_{imp}^{-1}$). So in this limit, we may ignore the effect of disorder under some mild assumptions (dilute density, no localization). Furthermore at $\mu = 0$ limit of interest to us, at zero temperature there will be no screening effect arising from the Dirac electrons. But at elevated temperature there can be thermally excited population of electrons and holes that may lead to classical (Debye) screening. Since our

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[1] note that in two space dimensions there is no γ^5 matrix and hence no chirality in the sese of left/right-handedness can be defined

kinetic theory formulation is second order in Coulomb interaction (α), consideration of screening in this case will correspond to higher order effects in α. Therefore to be consistent at order α^2, we ignore the screening effects even at elevated temperatures. So we can focus on the sole role of Coulomb interactions. It is important to note that the Coulomb interactions can not be covariantly written in terms of the above Gallilean transformations. This is because despite an emergent spacetime structure of the form (2), the photons mediating the Coulomb interaction mostly pass through the vacuum. Therefore, although some single particle properties like density of states at non-zero chemical potential can be obtained from the Jacobian of the above transformation, the many-body properties in the presence of Coulomb forces do not conform to this logic.

Investigation of the $\mu = 0$ tilted Dirac cone systems at non-zero temperature will allow us to ignore the impurities, thereby to study the effects that purely arise from the spacetime structure (2) combined with Coulomb interactions. The general outcome is that, the presence of tilt enhances the many-body effects as manifested in the broadening of the Drude peak, in agreement with earlier study using independent method [33]. In section II we introduce the Hamiltonian and notations. In section III we consider the collisionless limit of the Boltzmann equations for Dirac fermions in the background metric (2). In section IV we consider the effect of Coulomb interactions in such a spacetime background. We end the paper by discussions and outlook in section V.

II. HAMILTONIAN AND NOTATIONS

We study the kinetic theory of electrons in tilted Dirac cone materials with Coulomb interactions. The total Hamiltonian consists of two terms: the free part H_0 and the interaction term H_{Coulomb}. The free Hamiltonian of tilted Dirac electrons is given by

$$H_0 = \psi^{\dagger a} \begin{pmatrix} \vec{p} \cdot \vec{\zeta} & (p_x - ip_y) \\ (p_x + ip_y) & \vec{p} \cdot \vec{\zeta} \end{pmatrix} \psi^a, \qquad (4)$$

where a accouts for spin-valley index. The interaction Hamiltonian is assumed to be the Coulomb electron-electron interaction given by

$$H_{\text{Coulomb}} = \frac{1}{2} \int \frac{\mathrm{d}^2\vec{p_1}}{(2\pi)^2} \frac{\mathrm{d}^2\vec{p_2}}{(2\pi)^2} \frac{\mathrm{d}^2\vec{q}}{(2\pi)^2} \\ \times \psi^{\dagger a}_{\vec{p_1}-\vec{q}} \psi^a_{\vec{p_1}} V(q) \psi^{\dagger b}_{\vec{p_2}+\vec{q}} \psi^b_{\vec{p_2}}, \qquad (5)$$

where $V(q)$ is the static Coulomb potential

$$V(q) = \frac{2\pi e^2}{\epsilon |\vec{q}|}, \qquad (6)$$

and ϵ is the dielectric constant of the medium. The free Hamiltonian can be diagonalized through a unitary transformation $H_0^{\text{diag}} = U^\dagger H_0 U$. The transformation matrices are

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ e^{i\phi_p} & -e^{i\phi_p} \end{pmatrix}, \qquad (7)$$

where ϕ_p is the polar angle of vector $\vec{p}$. In the diagonal basis, the electron field is transformed into

$$\psi_{\vec{p}} = \begin{pmatrix} \psi_{1,\vec{p}} \\ \psi_{2,\vec{p}} \end{pmatrix} \rightarrow \psi_{\vec{p}}^{\text{diag}} = U^\dagger \psi_{\vec{p}} = \begin{pmatrix} c_{+,\vec{p}} \\ c_{-,\vec{p}} \end{pmatrix}, \qquad (8)$$

where

$$c_{+,\vec{p}} = \frac{1}{\sqrt{2}}(\psi_{1,\vec{p}} + \psi_{2,\vec{p}}), \\ c_{-,\vec{p}} = \frac{P}{|\vec{p}|} \frac{1}{\sqrt{2}}(\psi_{1,\vec{p}} - \psi_{2,\vec{p}}), \qquad (9)$$

and $P = p_x + ip_y$. Then, the diagonal free Hamiltonian is given by

$$H_0^{\text{diag}} = \sum_{a,s} \int \frac{\mathrm{d}^2\vec{p}}{(2\pi)^2} E_{s,\vec{p}} c^\dagger_{a,s,\vec{p}} c_{a,s,\vec{p}}, \qquad (10)$$

where

$$E_{s,\vec{p}} = s v_F |\vec{p}| + v_F \vec{p} \cdot \vec{\zeta}, \qquad (11)$$

and $s = \pm$ is the band index denoting upper ($+$) and lower ($-$) branches of the tilted Dirac cone, while as mentioned, a is the spin-valley index.

The interaction Hamiltonian in diagonal basis $c_{a,s}$ becomes

$$H_{\text{Coulomb}} = \sum_{s_1,s_2,s_3,s_4} \int \frac{\mathrm{d}^2\vec{p_1}}{(2\pi)^2} \frac{\mathrm{d}^2\vec{p_2}}{(2\pi)^2} \frac{\mathrm{d}^2\vec{q}}{(2\pi)^2} T_{s_1 s_2 s_3 s_4}(k_1, k_2, q) \\ \times c^\dagger_{b,s_4,\vec{p_1}+\vec{q}} c^\dagger_{a,s_3,\vec{p_2}-\vec{q}} V(q) c_{a,s_2,\vec{p_2}} c_{b,s_1,\vec{p_1}}, \qquad (12)$$

with

$$T_{s_1 s_2 s_3 s_4}(p_1, p_2, q) = \frac{1}{8} V(q) \times \left(1 + s_1 s_4 \frac{(P_1^* + Q^*)P_1}{|\vec{p_1} + \vec{q}||\vec{p_1}|} \right) \\ \times \left(1 + s_2 s_3 \frac{(P_2^* - Q^*)P_2}{|\vec{p_2} - \vec{q}||\vec{p_2}|} \right), \qquad (13)$$

where s_1, s_2 and s_3, s_4 are the band indices of incoming and outgoing particles, respectively and $Q = q_x + iq_y$. Moreover, electric current $J^i = -ev_F \bar{\psi}\gamma^i\psi$ in the diagonal basis is

$$\vec{J} = e\nu_F \sum_s \int \frac{\mathrm{d}^2\vec{p}}{(2\pi)^2} \left[\left(\vec{\zeta} + s\frac{\vec{p}}{|\vec{p}|} \right) c^\dagger_{s,\vec{p}} c_{s,\vec{p}} \right. \\ \left. + i\frac{\hat{z} \times \vec{p}}{|\vec{p}|} s c^\dagger_{s,\vec{p}} c_{-s,\vec{p}} \right], \qquad (14)$$

where $\hat{z}$ is the unit vector transverse to the xy plane of the material hosting the tilted Dirac fermions.

III. COLLISIONLESS LIMIT

To set the stage, in this section by turning off the Coulomb interactions, we study collisionless limit of the transport equation for the intraband and interband transition processes. To calculate electric transport coefficient, we apply an electric

field. The interaction Hamiltonian between the applied electric field and the tilted Dirac fermions is [34]

$$H_{\text{int}} = -e\vec{E} \cdot (i\psi_{\vec{p}}^{\text{diag}} \vec{\partial}\psi_{\vec{p}}^{\text{diag}})$$

$$= -e\vec{E} \cdot \sum_{a,s} \int \frac{\mathrm{d}^2\vec{p}}{(2\pi)^2} \Big(ic_{a,s,\vec{p}}^{\dagger} \vec{\partial} c_{a,s,\vec{p}} + \frac{1}{2} c_{a,s\vec{p}}^{\dagger} c_{a,-s,\vec{p}} \vec{\partial}\phi_{\vec{p}} \Big). \tag{15}$$

For later applications, we define the band diagonal (off-diagonal) distribution functions f (g) as follows

$$f_s(\vec{p}, t) = \langle c_{a,s,\vec{p}}^{\dagger} c_{a,s,\vec{p}} \rangle,$$
$$g_s(\vec{p}, t) = \langle c_{a,s,\vec{p}}^{\dagger} c_{a,-s,\vec{p}} \rangle. \tag{16}$$

There is no sum over spin-valley index a. We assume that these distribution functions are the same for all spin and valley indices. We use the Liouville equation [35]

$$\frac{\mathrm{d}\mathcal{O}}{\mathrm{d}t} = \frac{1}{i\hbar}[\mathcal{O}, H_0^{\text{diag}} + H_{\text{int}}], \tag{17}$$

to find the (generalized) Boltzmann equation for the distribution functions that correspond to intraband (f) and interband (g) transition as

$$\frac{\partial f_s(\vec{p}, t)}{\partial t} + e\vec{E} \cdot \vec{\partial}_p f_s(\vec{p}, t)$$
$$- i\frac{e\vec{E} \cdot \vec{\partial}_p \phi_p}{2|\vec{p}|}\Big(g_s(\vec{p}, t) - g_{-s}(\vec{p}, t) \Big) = 0, \tag{18}$$

and

$$\frac{\partial g_s(\vec{p}, t)}{\partial t} - i(E_{s,\vec{p}} - E_{-s,\vec{p}})g_s(\vec{p}, t) + e\vec{E} \cdot \vec{\partial}_p g_s(\vec{p}, t)$$
$$- i\frac{e\vec{E} \cdot \vec{\partial}_p \phi_p}{2|\vec{p}|}\Big(f_s(\vec{p}, t) - f_{-s}(\vec{p}, t) \Big) = 0. \tag{19}$$

Perturbative solution can be achieved by expanding the distribution functions as

$$f_s(p, t) = f_s^0(p, t) + f_s^1(p, t) + \cdots, \tag{20}$$
$$g_s(p, t) = g_s^0(p, t) + g_s^1(p, t) + \cdots. \tag{21}$$

At the leading order f_s is the Fermi-Dirac function

$$f_s^0(E_{s,\vec{p}}) = \frac{1}{1 + e^{\beta\gamma E_{s,\vec{p}}}}, \tag{22}$$

where $\gamma = (1 - \zeta^2)^{-1/2}$. Moreover, as can be seen from (19), at the leading order, $g_s(\vec{p}, s)$ is zero at equilibrium, namely $g_s^0(\vec{p}, t) = 0$, so at the first order the last term in (18) is vanishing. Thus, the distribution function f upto the first order is given by

$$f_s(\vec{p}, \omega) = 2\pi\delta(\omega)f^0(E_s) - e\vec{E} \cdot \vec{\partial}_p f^0(E_s)g(\vec{p}, \omega). \tag{23}$$

To compute the interband transition from Eq. (14) we need to find imaginary part of the distribution function $g_s(\vec{p}, t)$,

$$g_s(\vec{p}, t) = g_s'(\vec{p}, t) + ig_s''(\vec{p}, t). \tag{24}$$

To leading order, solutions to Eqs. (18) and (19) are

$$g_+''(\vec{p}, \omega) = \frac{ie\omega(f_+^0(\vec{p}) - f_-^0(\vec{p}))E_x(\omega)}{2|\vec{p}|(4v_F^2|\vec{p}|^2 - \omega^2)},$$
$$g(\vec{p}, \omega) = \frac{1}{-i\omega + \delta}, \tag{25}$$

where we have Fourier transformed from time to frequency with δ being an infinitesimal positive number. The electric field is applied along the x axis.

Finally, the electric transport coefficients for intraband transition are computed as follows:

$$\sigma_{\text{intra}}^{xx} = \frac{e^2}{h} \frac{2Nk_BT(1 - \zeta^2)^{\frac{1}{2}}\ln 2}{(-i\hbar\omega + \delta)\zeta^2}$$
$$\times \Big[-\frac{\zeta^2\zeta_x^2 - \zeta_y^2}{(1 - \zeta^2)^{\frac{3}{2}}} + \frac{1}{\zeta^2}\Big(1 + \frac{2\zeta^2 - 1}{(1 - \zeta^2)^{\frac{3}{2}}}\Big)(\zeta_x^2 - \zeta_y^2)\Big] \tag{26}$$

$$\sigma_{\text{intra}}^{xy} = \frac{e^2}{h} \frac{2Nk_BT(1 - \zeta^2)^{\frac{1}{2}}\ln 2}{(-i\hbar\omega + \delta)\zeta^2}$$
$$\times \Big[\frac{2}{\zeta^2}\Big(1 + \frac{2\zeta^2 - 1}{(1 - \zeta^2)^{\frac{3}{2}}}\Big) - \frac{1 + \zeta^2}{(1 - \zeta^2)^{\frac{3}{2}}}\Big]\zeta_x\zeta_y, \tag{27}$$

while for the interband transitions we have

$$\sigma_{\text{inter}}^{xx} = \frac{e^2N}{8h}M_1(a, \zeta, \alpha), \tag{28}$$

$$\sigma_{\text{inter}}^{xy} = -\frac{e^2N}{8h}M_2(a, \zeta, \alpha), \tag{29}$$

where

$$M_1[\bar{\omega}, \zeta, \alpha] = \int d\phi_p \Big[\frac{1}{1 + e^{\frac{\hbar\omega}{2k_BT}\gamma(-1 + \zeta\cos(\phi_p - \alpha))}}$$
$$- \frac{1}{1 + e^{\frac{\hbar\omega}{2k_BT}\gamma(1 + \zeta\cos(\phi_p - \alpha))}}\Big]\sin(\phi_p)^2, \tag{30}$$

$$M_2[\bar{\omega}, \zeta, \alpha] = \int d\phi_p \Big[\frac{1}{1 + e^{\frac{\hbar\omega}{2k_BT}\gamma(-1 + \zeta\cos(\phi_p - \alpha))}}$$
$$- \frac{1}{1 + e^{\frac{\hbar\omega}{2k_BT}\gamma(1 + \zeta\cos(\phi_p - \alpha))}}\Big]\sin(\phi_p)\cos(\phi_p) \tag{31}$$

Here $\bar{\omega} = \frac{\hbar\omega}{k_BT}$ and α is the angle of $\vec{\zeta}$ with respect to the x-axis. In the low frequency limit $\omega \to 0$, we have

$$M_1[a, \zeta, \alpha] = M_2[a, \zeta, \alpha] \to 0, \tag{32}$$

and interband contributions go to zero as they must.

IV. TRANSPORT IN PRESENCE OF COULOMB INTERACTION

After the warmup in section III with non-interacting tilted Dirac cone fermions, we are now ready to turn on the Coulomb interactions and investigate the fate of Coulomb interaction for tilted Dirac-cone fermions in order to find whether the Coulomb interactions play more important role in tilted Dirac fermions or not. In fact earlier investigation within the Fermi liquid theory shows that the Coulomb interactions play more important role for tilted Dirac fermions rather than

the non-tilted Dirac fermions [33]. Therefore it is useful to study the role of Coulomb interactions within the kinetic theory approach. In particular we will focus on the $\mu = 0$ and high enough temperatures to ensure that the disorder does not play a significant role, and the Coulomb forces will be the

major players [11, 12]. As can be seen in Eq. (32), in the low frequency limit interband transition can be ignored. Here we restrict ourselves to this regime. The Boltzmann equation include scattering terms induced by Coulomb interaction that are given by

$$
\begin{aligned}
\frac{\partial f_s(\vec{p},t)}{\partial t} + e\vec{E}\cdot\vec{\partial}_p f_s(\vec{p},t) = &-\frac{2\pi}{v_F}\int \frac{\mathrm{d}^2\vec{p}_1}{(2\pi)^2}\frac{\mathrm{d}^2\vec{q}}{(2\pi)^2} \\
&\Big(\delta(E_{s,\vec{p}} + E_{-s,\vec{p}_1} - E_{s,\vec{p}+\vec{q}} - E_{-s,\vec{p}_1-\vec{q}})R_1(\vec{p},\vec{p}_1,\vec{q})\times \\
&\times \Big[f_s(\vec{p},t)f_{-s}(\vec{p}_1,t)\Big(1 - f_s(\vec{p}+\vec{q},t)\Big)\Big(1 - f_{-s}(\vec{p}_1-\vec{q},t)\Big) - f_s(\vec{p}+\vec{q},t)f_{-s}(\vec{p}_1-\vec{q},t)\Big(1-f_s(\vec{p},t)\Big)\Big(1-f_{-s}(\vec{p}_1,t)\Big)\Big] \\
&- \delta(E_{s,\vec{p}} + E_{s,\vec{p}_1} - E_{s,\vec{p}+\vec{q}} - E_{s,\vec{p}_1-\vec{q}}))R_2(\vec{p},\vec{p}_1,\vec{q})\times \\
&\times \Big[f_s(\vec{p},t)f_s(\vec{p}_1,t)\Big(1 - f_s(\vec{p}+\vec{q},t)\Big)\Big(1 - f_s(\vec{p}_1-\vec{q},t)\Big) - f_s(\vec{p}+\vec{q},t)f_s(\vec{p}_1-\vec{q},t)\Big(1-f_s(\vec{p},t)\Big)\Big(1-f_s(\vec{p}_1,t)\Big)\Big]\Big),
\end{aligned}
\tag{33}
$$

where $R_1(\vec{p},\vec{p}_1,\vec{q})$ and $R_2(\vec{p},\vec{p}_1,\vec{q})$ are scattering amplitude functions for particles of opposite (i.e. electron-hole) and identical (i.e. electron-electron or hole-hole) charge, respec-

tively which are given in the appendix A. We linearize the distribution function $f_s(\vec{p},\omega)$ in Boltzmann equation (33) and use the ansatz (23) to obtain

$$
\begin{aligned}
\frac{\beta' v_F(-i\omega g(\vec{p},\omega) - 1)}{(1 + e^{\beta' E_{s,\vec{p}}})(1 + e^{-\beta' E_{s,\vec{p}}})}\Big(s\frac{\vec{p}}{p} + \vec{\zeta}\Big) = &-2\pi\int \frac{\mathrm{d}^2\vec{p}_1}{(2\pi)^2}\frac{\mathrm{d}^2\vec{q}}{(2\pi)^2} \\
&\Bigg[\frac{\delta(E_{s,\vec{p}} + E_{-s,\vec{p}_1} - E_{s,\vec{p}+\vec{q}} - E_{s,\vec{p}_1-\vec{q}})R_1(\vec{p},\vec{p}_1,\vec{q})}{(1 + e^{-\beta' E_{s,\vec{p}}})(1 + e^{-\beta' E_{-s,\vec{p}_1}})(1 + e^{\beta' E_{s,\vec{p}+\vec{q}}})(1 + e^{\beta' E_{-s,\vec{p}_1-\vec{q}}})}\times \\
&\times \Big(\frac{\partial E_{s,\vec{p}}}{\partial\vec{p}}g(\vec{p},\omega) + \frac{\partial E_{-s,\vec{p}_1}}{\partial\vec{p}_1}g(\vec{p}_1,\omega) - \frac{\partial E_{s,\vec{p}+\vec{q}}}{\partial(\vec{p}+\vec{q})}g(\vec{p}+\vec{q},\omega) - \frac{\partial E_{-s,\vec{p}_1-\vec{q}}}{\partial(\vec{p}_1-\vec{q})}g(\vec{p}_1-\vec{q},\omega)\Big) \\
&+ \frac{\delta(E_{s,\vec{p}} + E_{s,\vec{p}_1} - E_{s,\vec{p}+\vec{q}} - E_{s,\vec{p}_1-\vec{q}}))R_2(\vec{p},\vec{p}_1,\vec{q})}{(1 + e^{-\beta' E_{s,\vec{p}}})(1 + e^{-\beta' E_{s,\vec{p}_1}}(1 + e^{\beta' E_{s,\vec{p}+\vec{q}}})(1 + e^{\beta' E_{s,\vec{p}_1-\vec{q}}})}\times \\
&\times \Big(\frac{\partial E_{s,\vec{p}}}{\partial\vec{p}}g(\vec{p},\omega) + \frac{\partial E_{s,\vec{p}_1}}{\partial\vec{p}_1}g(\vec{p}_1,\omega) - \frac{\partial E_{s,\vec{p}+\vec{q}}}{\partial(\vec{p}+\vec{q})}g(\vec{p}+\vec{q},\omega) - \frac{\partial E_{s,\vec{p}_1-\vec{q}}}{\partial(\vec{p}_1-\vec{q})}g(\vec{p}_1-\vec{q},\omega)\Big)\Bigg],
\end{aligned}
\tag{34}
$$

where β' is defined as $\beta' = \beta\gamma$. Finding an analytic solution to the above integro-differential equation even in the linearized limit is formidable task. Therefore, we numerically solve the Boltzmann equation as discussed in the following.

There are points in the momentum space where the Coulomb interaction is singular and need to be handled with special care. These points are given by $\vec{q} \to 0$ and $\vec{q} \to \vec{p}_1 - \vec{p}$ where the Coulomb potential $V(\vec{q})$ and $V(\vec{p}_1 - \vec{p} - \vec{q})$ diverge as inverse square of the distance to 0 and $\vec{p}_1 - \vec{p}$, respectively. This is because the Coulomb matrix element appears as the square of the T-matrices defining $R_1(\vec{p},\vec{p}_1,\vec{q})$ and $R_2(\vec{p},\vec{p}_1,\vec{q})$ (see appendix A). However, if we expand functions in (34) around $\vec{q} = 0$ and $\vec{q} = \vec{p}_1 - \vec{p}$, these points are zeros of those functions. Given the angular integration, the first order term does not contribute to the integral, leaving behind a second order term in the numerator. Thus the integral is regular around these points and can be numerically inte-

grated. The other source of divergences, are the anti-collinear and collinear scattering processes, which are the characteristic feature of linear energy dispersion in 2D materials [36].

The Boltzmann equation for time reversal invariant interactions can be derived from the variational principles [2, 37] which is explored in more detail in appendix B. Generally, we can cast the linearized Boltzmann equation into the operator formulation as [12]

$$
|\phi\rangle = C\,|g\rangle,
\tag{35}
$$

where $|g\rangle$ defines the perturbed distribution function and C defines the collision processes and $|\phi\rangle$ defines external perturbations such as the electric field and/or temperature gradients [12]. To find the perturbed state $|g\rangle$, we need to find inverse of collision operator C^{-1}. Following [11], if λ_n are the eigenvalues of collision operator C, then C^{-1} has the fol-

lowin spectral representation:

$$C^{-1} = \sum_n \frac{1}{\lambda_n} |n\rangle \langle n|, \qquad (36)$$

where $|n\rangle$ is the n'th eigenvector of collision operator C. It is clear that the perturbed state $|g\rangle$ are dominated by the smallest eigenvalue of the collision operator C.

In the case of Coulomb interaction, the right hand side of Eq. (34) can be viewed as a linear operator acting on $\frac{\partial E_{s,\vec{p}}}{\partial \vec{p}} g(\vec{p}, \omega)$. So, one can define an inner product with respect to which the above operator is Hermitian as follows:

$$\langle g_1 | g_2 \rangle = \sum_s \int \frac{d^2 p}{(2\pi)^2} g_{1s}(\vec{p}) g_{2s}(\vec{p}). \qquad (37)$$

Based on this, we introduce a functional whose stationary solution are the Boltzmann Eq. (34) as follows:

$$\mathcal{Q}(g) = \frac{2\pi}{8} \int \frac{d^2 p}{(2\pi)^2} \frac{d^2 p_1}{(2\pi)^2} \frac{d^2 q}{(2\pi)^2} \left[\frac{\delta(E_{s,\vec{p}} + E_{-s,\vec{p}_1} - E_{s,\vec{p}+\vec{q}} - E_{-s,\vec{p}_1-\vec{q}}) R_1(\vec{p}, \vec{p}_1, \vec{q})}{(1 + e^{-\beta' E_{s,\vec{p}}})(1 + e^{-\beta' E_{-s,\vec{p}_1}})(1 + e^{\beta' E_{s,\vec{p}+\vec{q}}})(1 + e^{\beta' E_{-s,\vec{p}_1-\vec{q}}})} \times \right.$$
$$\times \frac{1}{v_F^2} \left(\frac{\partial E_{s,\vec{p}}}{\partial \vec{p}} \hat{g}(\vec{p}, \omega) + \frac{\partial E_{-s,\vec{p}_1}}{\partial \vec{p}_1} \hat{g}(\vec{p}_1, \omega) - \frac{\partial E_{s,\vec{p}+\vec{q}}}{\partial (\vec{p}+\vec{q})} \hat{g}(\vec{p}+\vec{q}, \omega) - \frac{\partial E_{-s,\vec{p}_1-\vec{q}}}{\partial (\vec{p}_1-\vec{q})} \hat{g}(\vec{p}_1-\vec{q}, \omega) \right)^2$$
$$+ \frac{\delta(E_{s,\vec{p}} + E_{s,\vec{p}_1} - E_{s,\vec{p}+\vec{q}} - E_{s,\vec{p}_1-\vec{q}})) R_2(\vec{p}, \vec{p}_1, \vec{q})}{(1 + e^{-\beta' E_{s,\vec{p}}})(1 + e^{-\beta' E_{s,\vec{p}_1}})(1 + e^{\beta' E_{s,\vec{p}+\vec{q}}})(1 + e^{\beta' E_{s,\vec{p}_1-\vec{q}}})} \times$$
$$\times \frac{1}{v_F^2} \left(\left(\frac{\partial E_{s,\vec{p}}}{\partial \vec{p}} \hat{g}(\vec{p}, \omega) + \frac{\partial E_{s,\vec{p}_1}}{\partial \vec{p}_1} \hat{g}(\vec{p}_1, \omega) - \frac{\partial E_{s,\vec{p}+\vec{q}}}{\partial (\vec{p}+\vec{q})} \hat{g}(\vec{p}+\vec{q}, \omega) - \frac{\partial E_{s,\vec{p}_1-\vec{q}}}{\partial (\vec{p}_1-\vec{q})} \hat{g}(\vec{p}_1-\vec{q}, \omega) \right)^2 \right]$$
$$\left. + \int \frac{d^2 p}{(2\pi)^2} \frac{\hat{g}(\vec{p}, \omega)(-\frac{i\omega \hat{g}(\vec{p}, \omega)}{2} - \beta' v_F)}{(1 + e^{\beta' E_{s,\vec{p}}})(1 + e^{-\beta' E_{s,\vec{p}}})} \left(1 + s \frac{\vec{p} \cdot \vec{\zeta}}{p} \right), \right.$$
$$(38)$$

where $\hat{g}(p, \omega) = \beta' v_F g(p, \omega)$.

Due to momentum conservation, the delta function of energy conservation of tilted Dirac matterial is the same as graphene. Delta function for energy conservation of same-charge particle scattering defines an ellipse for the set of points $\vec{q} = (q_x, q_y)$ where the sum of distances from two canonical points $\vec{p}_1$ and $-\vec{p}$ is constant, namely:

$$|\vec{p}| + |\vec{p}_1| - |\vec{p}+\vec{q}| - |\vec{p}_1-\vec{q}| = 0. \qquad (39)$$

We parameterize the ellipse by elliptic coordinates $0 \le \mu < \infty$ and $0 \le \theta < 2\pi$ in terms of which one can write [36]:

$$Q = \frac{P_1 - P}{2} + \frac{P_1 + P}{4}(e^{\mu+i\theta} + e^{-\mu-i\theta}), \qquad (40)$$

where $Q = q_x + iq_y$ and similarly for P and P_1. There is a constraint for delta function of same-charge particles as

$$\cosh(\mu) = \frac{p + p_1}{|p + p_1|}. \qquad (41)$$

For opposite-charge particles, the argument of the delta function of energy conservation defines a hyperbola for the set of points $\vec{q} = (q_x, q_y)$ where the differences of distances from two canonical points $\vec{p}_1$ and $-\vec{p}$ is constant, namely:

$$|\vec{p}| - |\vec{p}_1| - |\vec{p}+\vec{q}| + |\vec{p}_1-\vec{q}| = 0. \qquad (42)$$

Depending on the relative absolute value of p_1 and p, there is a similar constraint (e.g. for $p > p_1$)

$$\cos(\theta) = \frac{p - p_1}{|p + p_1|}. \qquad (43)$$

So we need to choose θ range appropriately to be consistent with $p > p_1$ (or $p_1 > p$). For the same-charge particle, we can integrate over momentum in the elliptic coordinate μ

$$\int \frac{d^2 \vec{q}}{(2\pi)^2} 2\pi \delta(\vec{p} + \vec{p}_1 + |\vec{p}+\vec{q}| - |\vec{p}_1-\vec{q}|) =$$
$$\int_0^{2\pi} \frac{d\theta}{2\pi} \frac{|\vec{p}+\vec{p}_1|}{4\sinh(\mu)} \left(\cosh^2(\mu) - \cos^2(\theta) \right). \qquad (44)$$

In the collinear limit $\vec{p} = (p, 0)$ and $\vec{p}_1 = (p_1, p_{1\perp})$, we find

$$\frac{|\vec{p}+\vec{p}_1|}{\sinh(\mu)} = (p + p_1)^2 \sqrt{\frac{p_1}{p}} \frac{1}{p_{1\perp}}. \qquad (45)$$

It is clear that integration over $p_{1\perp}$ is logarithmically divergent and the Coulomb interaction in the collinear limit can not cancel this singularity. This divergence is expected to be cutoff by higher order corrections to the self energy and will be proportional to structure constant α in the next order in the perturbation theory. The dominant contribution to the integral is in the range $\frac{T}{v_F}$ and $\frac{\alpha T}{v_F}$ [37]. Thus one estimates,

$$\int dp_{1\perp} \frac{1}{p_{1\perp}} \approx 2 \ln \alpha. \qquad (46)$$

To study the collinear limit in more detail, we look at linearized Boltzmann equation (34) for collision part of same-charge particles [9, 11]. For the collinear limit, we choose the momentum $\vec{p} = (p, 0)$ with $p > 0$ and further we write $\vec{p}_1 = (p_1, p_{1\perp})$, $\vec{q} = (q, q_\perp)$ where $p_{1\perp}$ and $q_\perp$ are small.

In the regime where $p > 0$, $p + q > 0$ and $p_1 - q > 0$, the argument of the energy conservation delta function can be simplified as:

$$
\begin{aligned}
&|\vec{p}| + |\vec{p}_1| - |\vec{p} + \vec{q}| - |\vec{p}_1 - \vec{q}| \\
&\approx \frac{p_{1\perp}^2}{2p_1} - \frac{q_\perp^2}{2(p+q)} - \frac{(p_{1\perp} - q_\perp)^2}{2(p_1 - q)} = \\
&- \frac{p_1 + p}{2(p+q)(p_1 - q)}(q_\perp - A_+ p_{1\perp})(q_\perp - A_- p_{1\perp})
\end{aligned}
\tag{47}
$$

where $A_\pm$ are defined by:

$$
A_\pm = \frac{p+q}{p+p_1} \pm \sqrt{\frac{p(p_1 - q)(p+q)}{p_1(p+p_1)^2}}.
\tag{48}
$$

So the contribution arising from $R_2(\vec{p}, \vec{p}_1, q)$ to the Boltzmann equation becomes:

$$
\begin{aligned}
&\approx \frac{8\alpha^2 (\ln\alpha)\nu_F}{\pi} \frac{\vec{p}}{p} \int dp_1 \frac{dq}{q^2} \sqrt{\frac{p_1(p+q)(p_1 - q)}{p}} \\
&\times \frac{g(p,\omega) + g(p_1,\omega) - g(p+q,\omega) - g(p_1 - q,\omega)}{(1 + e^{-\beta'\nu_F(p+p\zeta_x)})(1 + e^{-\beta'\nu_F(p_1+p_1\zeta_x)})(1 + e^{\beta'\nu_F(p+q+(p+q)\zeta_x)})(1 + e^{\beta'\nu_F(p_1-q-(p_1-q)\zeta_x)})},
\end{aligned}
\tag{49}
$$

where in the collinear limit, we have used $p > 0, p + q > 0$ and $p_1 - q > 0$. Exchange terms are negligible when $\alpha \to 0$ where a logarithm dominates and the Debye screening mass makes the above integral convergent in the $q \to 0$ limit [9].

As can be seen in Eq. (49), the same-charge collision integral that is proportional to $\alpha^2 \ln\alpha$ becomes zero if the momentum-independent ansatz $g(p,\omega) \propto g(\omega)$ is employed. However, when this ansatz is used, there will be other non-collinear (and hence non-divergent) terms proportional to α^2 (rather than $\alpha^2 \log\alpha$). These types of terms in the large logarithm limit contribute smaller eigenvalues and hence by Eq. (36) dominate the distribution function.

Hence up to logarithm corrections we are led to adopt the following ansatz:

$$
\hat{g}(p,\omega) = v_F \beta'^2 g(\omega),
\tag{50}
$$

where the coefficients are chosen so as to make $g(\omega)$ dimensionless. Finally, we integrate the functional $\mathcal{Q}[g]$ using the coordinate parametrization (40) to obtain

$$
\mathcal{Q}[g] = \gamma\beta' \frac{\ln 2}{4\pi}\left[\gamma^2 \kappa(\zeta)\alpha^2 g(\omega)^2 - i\omega\beta' g(\omega)^2 - 2g(\omega)\right].
\tag{51}
$$

It should be noted that the parameter κ depends on the absolute value of the tilt parameter and not its direction with respect to the electric field which is denoted by θ_t. We can show that this fact by reparametrization of theta angle θ_l by $\theta_l - \theta_t$ if we write the measure d^2l as $ldld\theta_l$, where $l = k, k_1$ or q. The function $\kappa(\zeta)$ has been numerically computed for different values of tilted parameters as follows (see appendix C):

ζ	0	0.2	0.4	0.6	0.8
κ	3.65	3.83	4.26	5.37	7.97

We have plotted the results in Fig. 1. The Drude pole of conductivity coefficients Eq. (54) is given by

$$
\omega = \frac{\kappa(\zeta)\alpha^2 k_B T}{i\hbar\sqrt{1 - \zeta^2}},
\tag{52}
$$

which shows that the frequency of absorption enhanced than graphene (i.e. $\zeta = 0$ Dirac fermions) by factor of the form $\propto (1 - \zeta^2)^{-\frac{1}{2}}$. This can be regarded as a direct manifestation of the underlying spacetime metric (2).

Finally, we extremize the functional $\mathcal{Q}[g]$ with respect to the $g(\omega)$ to find the perturbation ansatz (50) as

$$
\hat{g}(p,\omega) = \frac{\frac{v_F}{T}}{-i\omega\sqrt{1 - \zeta^2} + \kappa(\zeta)\alpha^2 T}.
\tag{53}
$$
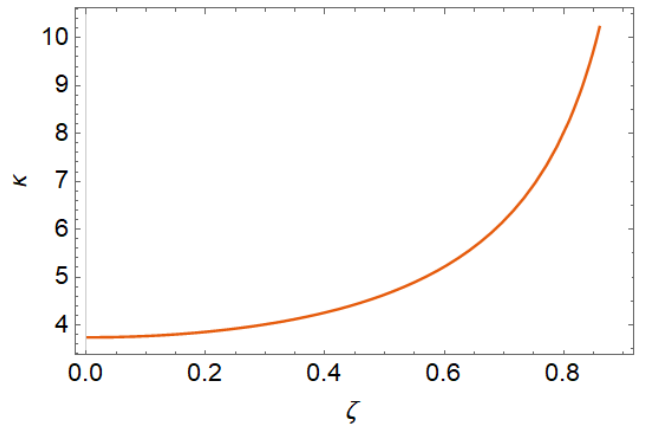
FIG. 1. Numerically computed values of κ as a function of ζ.
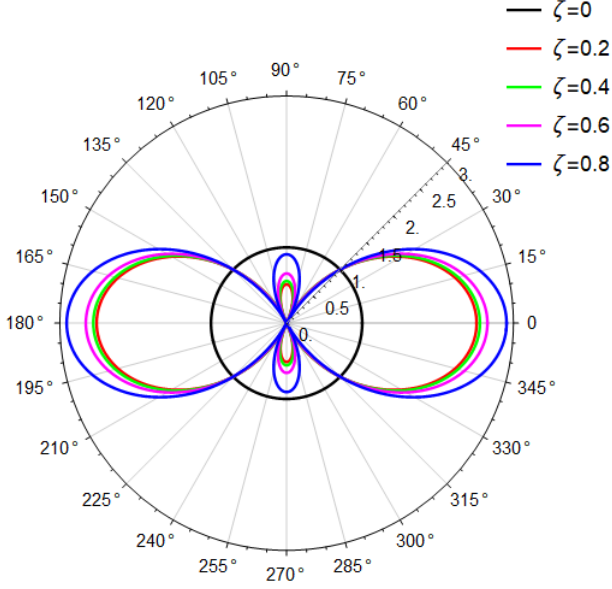
FIG. 2. Angular dependence of the dimensionless Drude pole strength χ.

Then, the intraband conductivity coefficients can be read off as follows

$$\sigma^{xx}_{\text{intra}} = \frac{e^2}{h} \frac{2Nk_BT \ln 2(1-\zeta^2)}{(-i\hbar\omega\sqrt{1-\zeta^2} + \kappa(\zeta)\alpha^2 k_BT)\zeta^2} \times$$
$$\times \left[-\frac{\zeta^2\zeta_x^2 - \zeta_y^2}{(1-\zeta^2)^{\frac{3}{2}}} + \frac{1}{\zeta^2}\left(1 + \frac{2\zeta^2 - 1}{(1-\zeta^2)^{\frac{3}{2}}}\right)(\zeta_x^2 - \zeta_y^2)\right], \tag{54}$$

$$\sigma^{xy}_{\text{intra}} = \frac{e^2}{h} \frac{2Nk_BT \ln 2(1-\zeta^2)}{(-i\hbar\omega\sqrt{1-\zeta^2} + \kappa(\zeta)\alpha^2 k_BT)\zeta^2} \times$$
$$\times \left[\frac{2}{\zeta^2}\left(1 + \frac{2\zeta^2 - 1}{(1-\zeta^2)^{\frac{3}{2}}}\right) - \frac{1+\zeta^2}{(1-\zeta^2)^{\frac{3}{2}}}\right]\zeta_x\zeta_y. \tag{55}$$

The above equations indicate that, the tilting of the Dirac cone increases the broadening Drude frequency by $\kappa(\zeta) \times \frac{1}{\sqrt{1-\zeta^2}}$.
In addition to that, the strength of the Drude pole is also enhanced. To quantify the effect of the tilt, let us define a dimensionless conductivity by

$$\chi = \frac{\text{Res}(\sigma^{xx}_\zeta)}{\text{Res}(\sigma^{xx}_{\zeta=0})}, \tag{56}$$

where the Res means the residue at the Drude pole and defiens the strength of the Drude pole. In Fig. 2 the polar plot of χ is shown. As is evident, the strength of the pole are enhanced 2~3 times for different values of ζ when angle between E_x and $\vec{\zeta}$ are not too large.

V. SUMMARY AND OUTLOOK

When electrons are in a spacetime with non-zero ζ, the time intervals measured in parts of spacetime that have zero ζ are

related to those in the spacetime (2) by

$$dt = \frac{dt_0}{\sqrt{1-\zeta^2}}. \tag{57}$$

This means that the corresponding frequencies are related by

$$\delta\omega_0 = \frac{\delta\omega}{\sqrt{1-\zeta^2}}. \tag{58}$$

Therefore, the part of the broadening in Eq. (52) that is proportional to $(1-\zeta^2)^{-1/2}$, is a manifestation of an underlying spacetime structure given in Eq. (2). The additional broadening contained in $\kappa(\zeta)$ (see Fig. 1) arise from particular form of the Coulomb integrals and can be attributed to the role of interaction in a background spacetime with non-zero tilt parameter ζ [26]. In addition to broadening of the Drude peak, the strength of the pole is also enhanced as given by Eq. (56) and plotted in Fig. 2. It is important to note that in this calculation we have not explicitly used the metric Eq. (2), nevertheless, the above redshift factors appear quite naturally, indicating that the optical process and the interactions are taking place in an underlying metric structure. The effects encoded in $\kappa(\zeta)$ can not be captured by a simple coordinate transformation. This is because the Coulomb forces (being a force between electrons in a material) can not be covariantly expressed in the above spacetime, as the Coulomb forces in a 2D material are mediated by photons that mostly propagate in 3D (Minkowski) spacetime outside the material.

Therefore the conductivity of tilted Dirac-cone fermions contains a wealth of information about the tilt parameter ζ. This parameter defines an underlying spacetime metric (2). If for a tilted Dirac cone system, one can enforce the tilt parameter ζ to vary in space over the length scales $\gtrsim \lambda$, where λ is the wavelength of the light, from the local measurement of the optical conductivity and only the width and strength of the Drude peak, one can reconstruct the $\zeta(\vec{x})$ from which the full structure of the emergent spacetime can be determined.

VI. ACKNOWLEDGMENTS

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Appendix A: Appendix A

The equilibrium distribution function in the covariant form is $f^0_s(p, E_s) = \frac{1}{1+e^{\beta U^\mu p_\mu}}$, where U^μ is the normalized fluid velocity ($U^\mu U_\mu = -1$) with respect to the metric (2) and $p_\mu = (E_s, \vec{p})$ is the energy-momentum four vector.

For $+$ charge particles, we have the following amplitude

functions:

$$
\begin{aligned}
&R_1(p, p_1, q) \\
&= 4\left[T_{+--+}(p, p_1, q) - T_{+-+-}(p, p_1, -p - q + p_1)\right|^2 \\
&+ 3|T_{+--+}(p, p_1, q)|^2 + 3|T_{+-+-}(p, p_1, -p - q + p_1)|^2\Big],
\end{aligned} \tag{A1}
$$

and

$$
\begin{aligned}
&R_2(p, p_1, q) \\
&= 4\left[\frac{1}{2}|(T_{++++}(p, p_1, q) - T_{++++}(p, p_1, p_1 - q - p))|^2\right. \\
&+ 3|T_{++++}(p, p_1, q)|^2 + 3|T_{++++}(p, p_1, p_1 - q - p)|^2\Big].
\end{aligned} \tag{A2}
$$

Appendix B: Appendix B

In this appendix, we look at the Boltzmann equation from variational principle in more detail. Our discussion in this section is based on the classic text of Ziman [2]. We consider elastic scattering of a particle from a state with momentum p to state p'. The probability of this scattering is given by:

$$
\mathcal{P}_p^{p'} = f_p(1 - f_{p'})\mathcal{D}_p^{p'}dp', \tag{B1}
$$

where $\mathcal{D}_p^{p'}$ is the intrinsic transition amplitude from state p to p' which can be computed from Feynman diagrams. For time reversal interactions, we have following condition:

$$
\mathcal{D}_p^{p'} = \mathcal{D}_{p'}^p. \tag{B2}
$$

The scattering part of this interaction can be summarized as follows:

$$
\sim \int \left(f_p(1 - f_{p'}) - f_{p'}(1 - f_p)\right)\mathcal{D}_p^{p'}dp'. \tag{B3}
$$

The linearized Collision part around the equilibrium distribution function f_p^0 is given by:

$$
\sim \int \left((f_{p'} - f_{p'}^0) - (f_p - f_p^0)\right)\mathcal{D}_p^{p'}dp'. \tag{B4}
$$

We assume that non-equilibrium distribution function is caused by turning on an electric field and a temperature gradient, so Boltzmann equation reads:

$$
\begin{aligned}
&-v_p.\frac{\partial f_p^0}{\partial T}\nabla T - v_p.e\frac{\partial f_p^0}{\partial \epsilon_p}\vec{E} \\
&= \int \left((f_{p'} - f_{p'}^0) - (f_p - f_p^0)\right)\mathcal{D}_p^{p'}dp',
\end{aligned} \tag{B5}
$$

where ϵ_p is the energy of the particles. It is convenient to write non-equilibrium distribution function as follows:

$$
f_p \equiv f_p^0 - \Phi_p\frac{\partial f_p^0}{\partial \epsilon_p}. \tag{B6}
$$

So Boltzmann equation is transformed to the following form:

$$
\begin{aligned}
&-v_p.\frac{\partial f_p^0}{\partial T}\nabla T - v_p.e\frac{\partial f_p^0}{\partial \epsilon_p}\vec{E} \\
&= \frac{1}{k_B T}\int (\Phi_p - \Phi_p')f_{p'}^0(1 - f_{p'}^0)\mathcal{D}_p^{p'}dp' \\
&= \frac{1}{k_B T}\int (\Phi_p - \Phi_p')\mathcal{P}_p^{p'}dp'.
\end{aligned} \tag{B7}
$$

We can generalized the above equation for collisions between particles in a time reversal interaction. For the scattering of the following form:

$$
\vec{p} + \vec{p'} = \vec{p''} + \vec{p'''}. \tag{B8}
$$

Similarly we have the intrinsic transition amplitude

$$
\mathcal{D}_{pp'}^{p''p'''}dp'dp''dp''' \tag{B9}
$$

that denotes the rate of scattering of particle with momentum $\vec{p}$ from a particle with momentum $\vec{p'}$ (in the range dp') into the particles with momentum p'' and p'''. The collision part in the Boltzmann equation is proportional to

$$
\begin{aligned}
&\int [f_p f_{p'}(1 - f_{p''})(1 - f_{p'''}) - f_{p''}f_{p'''}(1 - f_p)(1 - f_{p'})] \\
&\times \mathcal{D}_{pp'}^{p''p'''}dp'dp''dp'''.
\end{aligned} \tag{B10}
$$

Time reversal invariance of the interactions imposes

$$
\mathcal{D}_{pp'}^{p''p'''} = \mathcal{D}_{p''p'''}^{pp'}. \tag{B11}
$$

The equilibrium distribution function satisfies the relation

$$
f_p^0 f_{p'}^0(1 - f_{p''}^0)(1 - f_{p'''}^0) = f_{p''}^0 f_{p'''}^0(1 - f_p^0)(1 - f_{p'}^0), \tag{B12}
$$

and using the ansatz (B6) and expansion around the equilibrium distribution function f_p^0, collision term is simplified to

$$
\begin{aligned}
&\sim \frac{1}{k_B T}\int \left(\Phi_p + \Phi_{p'} - \Phi_{p''} - \Phi_{p'''}\right)\times \\
&f_p^0 f_{p'}^0(1 - f_{p''}^0)(1 - f_{p'''}^0)\mathcal{D}_{pp'}^{p''p'''}dp'dp''dp''' \\
&= \frac{1}{k_B T}\int \left(\Phi_p + \Phi_{p'} - \Phi_{p''} - \Phi_{p'''}\right)\times \\
&\mathcal{P}_{pp'}^{p''p'''}dp'dp''dp'''.
\end{aligned} \tag{B13}
$$

We can now explain variational principle for Boltzmann Eq. (B7) which can be summarized as an integral equation

$$
X(p) = \int (\Phi_p - \Phi_p')\mathcal{P}(p, p')dp'. \tag{B14}
$$

$X(p)$ in the left hand side of Eq. (B14) is a known function of temperature gradient or electric field. From the abstract point of view, Eq. (B14) has the following form:

$$
X = P\Phi. \tag{B15}
$$

This motivates to define an inner product as follows:

$$\langle \Phi_1 | \Phi_2 \rangle = \int dk \Phi_1(k) \Phi_2(k). \tag{B16}$$

With respect to this inner product, the scattering operator $\mathcal{P}(p, p')$ is Hermitian as follows:

$$\langle \Phi_1 | \mathcal{P}\Phi_2 \rangle = \langle \mathcal{P}\Phi_1 | \Phi_2 \rangle$$
$$= \frac{1}{2} \int (\Phi_p^1 - \Phi_{p'}^1) \mathcal{P}(p, p')(\Phi_p^2 - \Phi_{p'}^2) dp dp'. \tag{B17}$$

The most important property $\mathcal{P}$ is that it is positive definite

$$\langle \Phi | \mathcal{P}\Phi \rangle \geq 0, \tag{B18}$$

for all Φ.

Boltzmann Eq. (B15) implies that

$$\langle \Phi | \mathcal{P}\Phi \rangle = \langle \Phi | X \rangle. \tag{B19}$$

The variational principle states that, among all the functions Φ that satisfy Eq. (B19), the solution of Boltzmann Eq. (B15) is the one that corresponds to the maximum of Eq. (B18).

Appendix C: Appendix C

In this section we clarify how κ is numerically computed. The relevant part of $\mathcal{Q}(g)$ is as follows

$$\mathcal{Q}(g) = \frac{2\pi}{8} \int \frac{d^2p}{(2\pi)^2} \frac{d^2p_1}{(2\pi)^2} \frac{d^2q}{(2\pi)^2} \left[\frac{\delta(E_{s,\vec{p}} + E_{-s,\vec{p}_1} - E_{s,\vec{p}+\vec{q}} - E_{-s,\vec{p}_1-\vec{q}})R_1(\vec{p}, \vec{p}_1, \vec{q})}{(1 + e^{-\beta' E_{s,\vec{p}}})(1 + e^{-\beta' E_{-s,\vec{p}_1}})(1 + e^{\beta' E_{s,\vec{p}+\vec{q}}})(1 + e^{\beta' E_{-s,\vec{p}_1-\vec{q}}})} \times \right.$$
$$\times \frac{1}{v_F^2} \left(\frac{\partial E_{s,\vec{p}}}{\partial \vec{p}} \hat{g}(\vec{p}, \omega) + \frac{\partial E_{-s,\vec{p}_1}}{\partial \vec{p}_1} \hat{g}(\vec{p}_1, \omega) - \frac{\partial E_{s,\vec{p}+\vec{q}}}{\partial(\vec{p}+\vec{q})} \hat{g}(\vec{p}+\vec{q}, \omega) - \frac{\partial E_{-s,\vec{p}_1-\vec{q}}}{\partial(\vec{p}_1-\vec{q})} \hat{g}(\vec{p}_1-\vec{q}, \omega) \right)^2$$
$$+ \frac{\delta(E_{s,\vec{p}} + E_{s,\vec{p}_1} - E_{s,\vec{p}+\vec{q}} - E_{s,\vec{p}_1-\vec{q}}))R_2(\vec{p}, \vec{p}_1, \vec{q})}{(1 + e^{-\beta' E_{s,\vec{p}}})(1 + e^{-\beta' E_{s,\vec{p}_1}})(1 + e^{\beta' E_{s,\vec{p}+\vec{q}}})(1 + e^{\beta' E_{s,\vec{p}_1-\vec{q}}})} \times$$
$$\left. \times \frac{1}{v_F^2} \left((\frac{\partial E_{s,\vec{p}}}{\partial \vec{p}} \hat{g}(\vec{p}, \omega) + \frac{\partial E_{s,\vec{p}_1}}{\partial \vec{p}_1} \hat{g}(\vec{p}_1, \omega) - \frac{\partial E_{s,\vec{p}+\vec{q}}}{\partial(\vec{p}+\vec{q})} \hat{g}(\vec{p}+\vec{q}, \omega) - \frac{\partial E_{s,\vec{p}_1-\vec{q}}}{\partial(\vec{p}_1-\vec{q})} \hat{g}(\vec{p}_1-\vec{q}, \omega) \right)^2 \right], \tag{C1}$$

which consists of two terms, opposite charge and same charge scattering, which respectively are given as follows

$$\mathcal{I}_1(g) = \frac{2\pi}{8} \int \frac{d^2p}{(2\pi)^2} \frac{d^2p_1}{(2\pi)^2} \frac{d^2q}{(2\pi)^2} \left[\frac{\delta(E_{s,\vec{p}} + E_{-s,\vec{p}_1} - E_{s,\vec{p}+\vec{q}} - E_{-s,\vec{p}_1-\vec{q}})R_1(\vec{p}, \vec{p}_1, \vec{q})}{(1 + e^{-\beta' E_{s,\vec{p}}})(1 + e^{-\beta' E_{-s,\vec{p}_1}})(1 + e^{\beta' E_{s,\vec{p}+\vec{q}}})(1 + e^{\beta' E_{-s,\vec{p}_1-\vec{q}}})} \times \right.$$
$$\left. \times \frac{1}{v_F^2} \left(\frac{\partial E_{s,\vec{p}}}{\partial \vec{p}} \hat{g}(\vec{p}, \omega) + \frac{\partial E_{-s,\vec{p}_1}}{\partial \vec{p}_1} \hat{g}(\vec{p}_1, \omega) - \frac{\partial E_{s,\vec{p}+\vec{q}}}{\partial(\vec{p}+\vec{q})} \hat{g}(\vec{p}+\vec{q}, \omega) - \frac{\partial E_{-s,\vec{p}_1-\vec{q}}}{\partial(\vec{p}_1-\vec{q})} \hat{g}(\vec{p}_1-\vec{q}, \omega) \right)^2 \right], \tag{C2}$$

$$\mathcal{I}_2(g) = \frac{2\pi}{8} \int \frac{d^2p}{(2\pi)^2} \frac{d^2p_1}{(2\pi)^2} \frac{d^2q}{(2\pi)^2} \frac{\delta(E_{s,\vec{p}} + E_{s,\vec{p}_1} - E_{s,\vec{p}+\vec{q}} - E_{s,\vec{p}_1-\vec{q}}))R_2(\vec{p}, \vec{p}_1, \vec{q})}{(1 + e^{-\beta' E_{s,\vec{p}}})(1 + e^{-\beta' E_{s,\vec{p}_1}})(1 + e^{\beta' E_{s,\vec{p}+\vec{q}}})(1 + e^{\beta' E_{s,\vec{p}_1-\vec{q}}})} \times$$
$$\times \frac{1}{v_F^2} \left((\frac{\partial E_{s,\vec{p}}}{\partial \vec{p}} \hat{g}(\vec{p}, \omega) + \frac{\partial E_{s,\vec{p}_1}}{\partial \vec{p}_1} \hat{g}(\vec{p}_1, \omega) - \frac{\partial E_{s,\vec{p}+\vec{q}}}{\partial(\vec{p}+\vec{q})} \hat{g}(\vec{p}+\vec{q}, \omega) - \frac{\partial E_{s,\vec{p}_1-\vec{q}}}{\partial(\vec{p}_1-\vec{q})} \hat{g}(\vec{p}_1-\vec{q}, \omega) \right)^2 \right]. \tag{C3}$$

We first compute on $\mathcal{I}_2(g)$ which is simpler than $\mathcal{I}_1(g)$.

1. Like charge particle scattering

By the following transformations $\vec{k} = \beta' v_F \vec{p}$, $\vec{k}_1 = \beta' v_F \vec{p}_1$ and $\vec{q'} = \beta' v_F \vec{q}$, the like charge particle scattering term $\mathcal{I}_2$ is simplified as follows

$$\mathcal{I}_2(g) = \frac{\beta'}{64\pi^3} \alpha^2 g^2(\omega) \mathcal{I}_2'(g) \tag{C4}$$

where we have used explicit form of energy-momentum dispersion relation $E_{s,\vec{p}} = s v_F p + v_F \vec{p}.\zeta$, $\hat{g} = v_F \beta'^2 g(\omega)$, $\alpha = \frac{e^2}{v_F \epsilon}$ and $\mathcal{I}_2'$ is given by

$$\mathcal{I}_2'(g) = \int d^2k d^2k_1 d^2q' \frac{\delta(k + k_1 - | k + q' | - | k_1 - q' |)R_2'(\vec{k}, \vec{k}_1, \vec{q}')}{(1 + e^{-k + \vec{k}.\vec{\zeta}})(1 + e^{-k_1 + \vec{k}_1.\vec{\zeta}})(1 + e^{|k + q'| + (\vec{k} + \vec{q}').\vec{\zeta}})(1 + e^{|k_1 - q'| + (\vec{k}_1 - \vec{q}').\vec{\zeta}})}$$

$$\times \left(\frac{\vec{k}}{k} + \frac{\vec{k}_1}{k_1} - \frac{\vec{k} + \vec{q}}{| k + q |} - \frac{\vec{k}_1 - \vec{q}}{| k_1 - q |} \right)^2 \tag{C5}$$

where the difference between $R'(\vec{k}, \vec{k}_1, \vec{q}')$ and $R'(\vec{p}, \vec{p}_1, \vec{q})$ is that in the former $V(q')$ is $V(q') = \frac{1}{q'}$.

Using the parametrization (40), the Dirac delta function can be eliminated as follows

$$\mathcal{I}_2'(g) = \int d^2k d^2k_1 \int d^2q' \delta(k + k_1 - | k + q' | - | k_1 - q' |)...$$

$$= \int d^2k d^2k_1 \int_0^{2\pi} d\theta \int_0^\infty d\mu \frac{| k + k_1 |^2}{4} (cosh(\mu)^2 - cos(\theta)^2)\delta(k + k_1 - cosh(\mu) | k + k_1 |)...$$

$$= \int d^2k d^2k_1 \int_0^{2\pi} d\theta \frac{| k + k_1 | (cosh(\mu)^2 - cos(\theta)^2)}{4sinh(\mu)}... \tag{C6}$$

where ... are other functions in the integral. Finally $\mathcal{I}_2(g)$ part is summarized as follows

$$\mathcal{I}_2(g) = \frac{\beta'}{4 \times 64\pi^3} \alpha^2 g^2(\omega) \mathcal{I}_2'' \tag{C7}$$

$$\mathcal{I}_2''(g) = \int d^2k d^2k_1 \int_0^{2\pi} \frac{| k + k_1 | (cosh(\mu)^2 - cos(\theta)^2)}{sinh(\mu)}... \tag{C8}$$

2. Opposite charge particle scattering

The rest of the computation is the same as like charge particle scattering besides the fact that the argument of delta function is a hyperbola rather than an elliptic curve. We have the same relation for $\mathcal{I}_1(g)$ similar to eq. (C4) as follows :

$$\mathcal{I}_1(g) = \frac{\beta'}{64\pi^3} \alpha^2 g^2(\omega) \mathcal{I}_1'(g) \tag{C9}$$

and $\mathcal{I}_1'(g)$ similarly is given by

$$\mathcal{I}_1'(g) = \int d^2k d^2k_1 \int d^2q' \delta(k - k_1 - | k + q' | + | k_1 - q' |)...$$

$$= \int d^2k d^2k_1 \int_0^{2\pi} d\theta \int_0^\infty d\mu \frac{| k + k_1 |^2}{4} (cosh(\mu)^2 - cos(\theta)^2)\delta(k - k_1 - cos(\theta) | k + k_1 |)...$$

$$= 2 \int d^2k d^2k_1 \int_0^\infty d\mu \frac{| k + k_1 | (cosh(\mu)^2 - cos(\theta)^2)}{4sin(\theta)}..., \tag{C10}$$

the argument of the Dirac delta function have two solution for $cos(\theta) = \frac{k - k_1}{|k + k_1|}$ in the range $0 < \theta < 2\pi$ and prefactor 2 is the origin of it. To be consistent with $\mathcal{I}_2''$ part, we define $\mathcal{I}_1''(g)$ as $\mathcal{I}_1''(g) = 4\mathcal{I}_1'(g)$. The total contribution to $\mathcal{I}(g)$ is given by

$$\mathcal{I}(g) = \mathcal{I}_1(g) + \mathcal{I}_2(g) = (\frac{\gamma^3 \beta' log2}{4\pi})(\frac{\mathcal{I}_1''(g) + \mathcal{I}_2''(g)}{64\pi^2 log2\gamma^3})\alpha^2 g(\omega)^2,$$

$$\equiv (\frac{\gamma^3 \beta' log2}{4\pi})\kappa\alpha^2 g^2(\omega). \tag{C11}$$

The integrals $\mathcal{I}_1''(g)$ and $\mathcal{I}_1''(g)$ are convergent and can be integrated numerically by mathematica.

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