Dear Editor,

We thank the referee for their report on our manuscript *Bethe Ansatz, Quantum Circuits, and the F-basis* with pre-pint link scipost_202411_00037v1. Below, we address the referee's comments and the corresponding revisions.

Additionally, we have fixed typos, eliminated redundancies, updated the references, and improved the readability of the manuscript. A summary of the most significant changes introduced in the new version follows our response to the referee. The relocation of equation (3.23) to (3.31), accounted among the changes, has shifted the numbering of equations on pages 15–17. To avoid confusion, we refer to the labels of equations in the new version in this response unless otherwise stated.

Response to the referee

1. **Reviewer**: 'Above Eq (3.27), the authors state "(3.25) realizes a Bethe wave functions with M magnons propagating over N spins the inhomogeneous spin chain, we identify it the...". In Section 3.3, they state "Having obtained the MPS of the CBA (3.25), we are in the position to construct ABCs for the inhomogeneous spin chain..." It appears that Eq. (3.25) is crucial. Since the Bethe state should be an eigenstate of the inhomogeneous spin chain, could the authors provide some numerical verification of this equation for small lattice sizes?'

   We agree with the referee in that we have not adequately highlighted the role of (3.24) (equation (3.25) of the previous version).

   The importance of the MPS representation of the Bethe state in (3.24) is its ability to generate Bethe states with $0 \leqslant r \leqslant M$ magnons and support on $1 \leqslant k \leqslant N$ qubits by modifying its initialization in the auxiliary space. This property follows directly from (3.27) (equation (3.28) of the previous version), as demonstrated at the end of Subsection 3.1 for the homogeneous XXZ model. This result stems from the symmetry of the tensors of the MPS with respect to the exchange of qubits in the auxiliary space in the F-basis, combined with their convenient normalization. It naturally leads to the representation of Bethe states as linear superpositions of plane waves, as proved in Sub-appendix A.2. However, it is the property (3.27), rather than the explicit linear superposition of plane waves in (3.24), that is fundamental to the construction of the unitaries.

   Moreover, (3.27) is independent of whether the Bethe state in (3.24) is an eigenstate of the transfer matrix that defines the inhomogeneous spin chain. Ensuring that the Bethe state is an eigenstate requires imposing the Bethe equations on the spectral parameters u_a, or equivalently, on the quasi-momenta $p_{a,j}$.

   To improve clarity, we have introduced the following changes in the manuscript.

   The paragraph below (3.25) (equation (3.26) of the previous version) in page 16 now begins with the following sentence.

   - We assume that u_a, and consequently $p_{a,j}$, do not necessarily satisfy the Bethe equations. This means that the Bethe state (3.25) is not inevitably an eigenstate of the transfer matrix that defines the inhomogeneous spin chain.

We have also added the following paragraph below equation (3.27) in page 16.

- We emphasize that the importance of (3.27), alongside (3.20), lies in the fact that, by modifying the initialization of the MPS in the auxiliary space, the MPS also realizes Bethe states with $0 \leqslant r \leqslant M$ magnons and support on $1 \leqslant k \leqslant N$ qubits. This property is closely related to the equality between the MPS and the linear superpositions of plane waves of the CBA in (3.24).

Finally, Section 3.3 in page 16 now begins with the following sentence.

- Having obtained the MPS of the CBA (3.24) with the central property (3.27), we are in position to construct ABCs for the inhomogeneous spin chain.

We do not include numerical verification of the eigenvalue equation over solutions to the Bethe equations because, as we have explained, it is not relevant for our work.

2. **Reviewer**: ‘Above Table 2, the authors state: “However, this knowledge does not mean one can evaluate scalar products efficiently in general; rather, the limitation must be taken into account in the numerical computation of the unitaries.” Could they elaborate further on what this ”limitation” refers to and how it should be addressed? Additionally, the tables should be made clearer. For instance, what do “domain” and “image” mean in Tables 1-3?’

   We agree with the referee in that we should clarify the limitations of the method. We have updated the paragraph above Table 2 in page 22. We cite the paragraph below.

   - The reason for using (3.52) to compute closed formulae is that we know the entries exactly, thanks to the knowledge of the Bethe states (3.47). However, this knowledge implies that neither the scalar products among Bethe states nor the associated Gram matrices can be evaluated efficiently in general. The scalar product between two Bethe states with r magnons over k qubits involves the sum of $D = \binom{k}{r}$ distinct terms. Given k and r, the Gram matrix contains $(1/2)D(D+1)$ linearly independent scalar products. Gram matrices (for long unitaries) must be computed for each $1 \leqslant k \leqslant N - M$ and $0 \leqslant r \leqslant M$. The cost of these computations rapidly increases with N and M on a classical computer. Moreover, storing all the Gram matrices (or, equivalently, their numerical Cholesky factorization, which scales cubically with their size) demands an increasingly unfeasible amount of memory. These limitations must be taken into account in the classical numerical computation of the unitaries. Nonetheless, they do not diminish the usefulness of exact formulae for the unitaries in quantum computing. Exact formulae provide valuable data about the unitaries and, in particular, about their decomposition into one- and two-qubit unitaries.

   We have also changed the titles of the tables to make them clearer. Instead of ‘domain’ and ‘image’, we now use ‘input’ and ‘output’, respectively. For instance, now Table 1 reads as follows.

$P_j^{[i,r]}$	position	input	output	# rows × # columns	formulae
long	$1 \leqslant j \leqslant N-M$	$\mathsf{H}_M^{[r]}$	$\mathsf{H}_M^{[r-i]}$	$\binom{M}{r-i} \times \binom{M}{r}$	(3.44)
short	$N-M < j \leqslant N-1$	$\mathsf{H}_{N-j+1}^{[r]}$	$\mathsf{H}_{N-j}^{[r-i]}$	$\binom{N-j}{r-i} \times \binom{N-j+1}{r}$	(3.71)

Table 1: Features of the non-unitary building blocks $P_j^{[i,r]}$ of the unitaries of ABCs.

3. **Reviewer**: 'In Section 3, the number of qubits in which the unitaries act defines two classes: long and short unitaries, and they summarize the properties of long and short unitaries. So, what will happen if the number of spins N becomes infinite? It would be nice to provide some physical interpretation for this.'

   We agree with the referee that the limit $N \rightarrow \infty$ is an interesting question. However, we believe that the Bethe states prepared by the circuit do not provide the most suitable framework to address this issue. The reason is that the thermodynamic Bethe Ansatz, which is based on the density of Bethe roots, rather than the Algebraic or Coordinate Bethe Ansätze, used to construct Bethe states, encapsulates the dynamical features of the infinite spin chain.

   To address this point, we have added the footnote 7 in page 20, which we reproduce below.

   - One may wonder if any simplification occurs in the large N limit. However, Bethe states prepared by ABCs are not suited for addressing the thermodynamic limit. Instead, the thermodynamic Bethe Ansatz [52,53], based on the density of Bethe roots, rather than the ABA or the CBA for constructing Bethe states at finite N, is the proper approach for analyzing the thermodynamic limit of integrable models. At any rate, potential connections between ABCs and the thermodynamic Bethe Ansatz in the large N limit may be worth exploring.

   References [52] is the classic book on the thermodynamic Bethe Ansatz.

   - M. Takahashi, *Thermodynamics of One-Dimensional Solvable Models*, Cambridge University Press (1999), 10.1017/cbo9780511524332.

   Reference [53] is a more recent review on the thermodynamic Bethe Ansatz.

   - S.J. van Tongeren, "Introduction to the thermodynamic Bethe ansatz", *J. Phys. A* **49** (2016) 323005 [1606.02951].

Other changes

1. We have replaced 'pseudo-unitarity' and 'pseudo-unitary' with 'braided unitarity' and 'braided unitary', respectively, to align with the nomenclature used in the literature.

2. We have replaced 'ancillae' with 'post-selected qubits' when referring to the qubits removed in the quantum circuit of the ABCs. This change prevents confusion with the term 'ancillae', which we use to denote the qubits of the auxiliary space in the MPS.

3. We have moved equation (3.23) from page 15 of the previous version to page 17, where it now appears as (3.31). Additionally, we have added a new paragraph to clarify its role. This change improves the logical flow of the text. The new placement of the equation enhances the understanding of how Bethe states are arranged into eigenspaces of definite total spin along the z-axis from the perspective of the MPS formulation of the CBA.

   The paragraph we added is the following.

   - The arrangement of Bethe states into eigenspaces of definite total spin along the z-axis is clarified by the MPS of the CBA. To see this, it is convenient to assemble both Λ^0 and Λ^1 into the non-unitary tensor

     $$\Lambda_j = \Lambda_j(v_j; u_1, \ldots, u_M) : \mathsf{H}_M \otimes \mathsf{h}_j \cong \mathsf{H}_{M+1} \to \mathsf{H}_{M+1} \cong \mathsf{h}_j \otimes \mathsf{H}_M \ , \quad \Lambda_j^i := \langle i|_j \, \Lambda_j \, |0\rangle_j \ ,$$

     where $|i\rangle_j$ belongs to the Hilbert space of the j-th spin. We have defined Λ_j so that it moves the position of h_j in the tensor product from the last to the first place, which is convenient for the derivation of ABCs. Note we have not specified $\Lambda_j|1\rangle_j$ as it plays no role in the MPS. The tensor Λ_j by construction commutes and the total spin along z-axis of $M+1$ qubits (M ancillae and the j-th spin of the quantum space). Since the product of tensors in the auxiliary space of the MPS (3.24) is initialized on $|1\rangle^{\otimes M}$ and projected on $\langle 0|^{\otimes M}$, the Bethe state must carry a definite number of ones. We depict the MPS in terms of Λ_j in Figure 12. Note we have not specified $\Lambda_j|1\rangle_j$ as it plays no role in the MPS.

4. We have removed Figure 12. The figure was incorrect, as it would relate (3.31) with (2.33) like

   $$\Lambda_j = V_j^{-1} \widetilde{\mathscr{T}}_j V_{j-1} \ .$$

   This contradicts the freedom in choosing $\Lambda_j|1\rangle_j$, which is correlated with the freedom in $P_j|1\rangle_j$ to ensure the unitarity of long unitaries. Moreover, we believe that even a corrected version of Figure 12 would be unnecessary, as the relation between Λ_j^i and $\widetilde{\mathscr{T}}_j^i$ is already clear from (3.22).

5. Reference [23], which was previously listed as 'to appear', has been updated to the following reference.

   - R. Ruiz, A. Sopena, B. Pozsgay, E. López, "Efficient Eigenstate Preparation in an Integrable Model with Hilbert Space Fragmentation", 2411.15132.

We hope that the paper is suitable for publication with the aforementioned changes.

Kind regards,

Roberto Ruiz, Alejandro Sopena, Esperanza López, Germán Sierra, and Balázs Pozsgay