Response to the Referee 3

Strengths

1. Interesting numerical observation about operator entanglement in open quantum systems with strong symmetry
2. Clear and well presented

We thank the Referee for the positive assessment of our work about the interesting numerical observations and the presentation of our manuscript.

Weaknesses

1. Lack of analytical insights

We thank the Referee for raising this point. However, we don't think this would reduce the quality of our manuscript and affect the corresponding fulfillment of the SciPost Physics acceptance criterion, as we clarified below.

Report

Inspired by the results of Ref. [38] for an open quantum system with U(1) 'strong symmetry', the author studies the evolution of operator entanglement in a spin chain with SU(2) 'strong symmetry' under Lindblad dynamics. The model is a Heisenberg spin chain, with incoherent nearest-neighbor projections on singlets, which preserve the SU(2) symmetry. The initial state is chosen such that it is SU(2) symmetric (except in Section 5.4, where the initial state breaks the SU(2) symmetry, and is only U(1)-symmetric). The model is thoroughly studied numerically.

The main observation is that the operator entanglement undergoes an initial linear growth, followed by a decrease caused by decoherence, and a slow logarithmic growth at late times. This is the same phenomenology as in the U(1) case of Ref. [38], however, as emphasized by the author, an important difference with respect to the U(1) case is that the operator entanglement at long times does not come exclusively from the Shannon entropy of the number fluctuations between the to subsystems. This is reflected in the 'symmetry-resolved' operator entanglement which no longer vanishes, contrary to what happened in the U(1) case of Ref. [38].

The numerical results are presented in a clear way and are well discussed. Unfortunately, not much analytical insights are given, and the main observation of the paper remains unexplained.

Yet, I think the results are interesting enough, and could motivate further work in that direction. Therefore, in my opinion, it meets the Scipost Physics acceptance criterion 'Detail a groundbreaking theoretical/experimental/computational discovery'. I am not sure it meets any of the other criteria though.

We thank the Referee for the careful reading of our manuscript, especially for the positive assessment that "*The numerical results are presented in a clear way and are well discussed*" and "*I think the results are interesting enough, and could motivate further work in that direction.*" The Referee mainly concerns with the lack of analytical insights. However, we would like to mention that in the U(1)-symmetric case with dephasing considered in [Wellnitz *et al.*, PRL 129, 170401 (2022)], the analytical understanding is possible since the late-time operator entanglement in this case can be fully captured by the classical Shannon entropy associated with the probabilities for the half system being in different U(1) sectors, which is further related to the well-known classical symmetric simple exclusion process by the well-established perturbation theory in the strong dephasing limit, thus significantly simplifying the task of analytical understanding. In our SU(2)-symmetric case, both the classical Shannon entropy of probabilities and the symmetry-resolved operator entanglement have nontrivial contribution to the late-time operator entanglement, which makes the analytical understanding extremely hard. Moreover, since our model contains dissipation proportional to the dipole interaction between neighbor sites, even in the strong dissipation limit it is still hard to identify the density matrix ρ satisfying $\mathcal{L}_0(\rho) \equiv \sum_i (L_i \rho L_i^\dagger - \{L_i^\dagger L_i, \rho\}/2) = 0$ and the corresponding projector $\mathcal{P}$ onto these states, which makes the usual perturbation theory also unrealistic for our model. Therefore, for the above reasons we do not have analytical insights for our present results. However, as the Referee pointed out that "*the results are interesting enough,*" it would be interesting to further develop new theoretical tools or construct much simpler models with the same symmetry to achieve an analytical understanding of the results presented in this manuscript, which is left for future studies.

We would like to clarify that although we do not have an analytical understanding in this manuscript, we don't think this would reduce the quality of our manuscript, which indeed fulfills the SciPost Physics acceptance criterion. In addition to the criterion "Detail a groundbreaking theoretical/experimental/computational discovery" mentioned by the Referee, our work also established a synergetic link between the quantum information theory, open quantum many-body dynamics, and non-Abelian symmetries. While the operator entanglement, an important concept in quantum information theory to analyze the entangling capabilities of quantum evolution, has been used to study the closed nonequilibrium quantum many-body physics like quantum chaos and information scrambling very recently (see Refs. [53-55] of our manuscript), the application to the open quantum many-body systems is quite rare, especially its interplay with the symmetries. Although the Abelian U(1)-symmetric case with dephasing has been considered in [Wellnitz *et al.*, PRL 129, 170401 (2022)], it was our work that investigated for the first time the influence of the much more complicated non-Abelian symmetry on the operator entanglement dynamics of open quantum many-body systems and obtained interesting new results. Therefore, our work indeed also satisfies the SciPost Physics acceptance criterion that "Provide a novel and synergetic link between different research areas" and is suitable for publication in SciPost Physics. We hope the Referee will also agree with this.

Requested changes

1. The 'symmetry-resolved' operator entanglement should be defined carefully. The author gives a very brief discussion of that definition, and cites a number of references (many of which are about state entanglement, not operator entanglement, which might cause confusion), as if the author believed that there is a unique way of defining 'symmetry-resolved operator entanglement'. Let me stress that this is not the case. Here the author uses a definition similar to the one of Ref. [38], but this choice should be discussed and justified. Other definitions of 'symmetry-resolved operator entanglement' are being used in the literature, for instance in Ref. [59], that do not match the one of Ref. [38].

We thank the Referee for raising this point. We would like to note that the definition of symmetry-resolved operator entanglement in Ref. [59] is actually the same as the one introduced in our manuscript and Ref. [38]. To the best of our knowledge, this is the only way to define symmetry-resolved operator entanglement, that is, the symmetry-resolved operator entanglement in a specific symmetry sector q of the global symmetry Q is defined through the renormalized Schmidt coefficients $\hat{\lambda}_{q,i_q} = \lambda_{q,i_q}/\sqrt{p_q}$ belonging to that sector using the usual definition of entanglement entropy, where i_q labels different Schmidt coefficient in the symmetry sector q and $p_q = \sum_{i_q} \lambda^2_{q,i_q}$ such that $\sum_{i_q} \hat{\lambda}^2_{q,i_q} = 1$ as required in the usual definition of entanglement entropy. To avoid any possible confusion, in this resubmission we have added an explicit definition of the symmetry-resolved operator entanglement as follows:

"It is useful to also introduce the symmetry-resolved operator entanglement, which has attracted extensive attention recently [56-61]. Specifically, following in Ref. [38], for a generic global symmetry Q, the operator entanglement for the MPDO decomposition (2) at certain bond can be split into different symmetry sectors, similarly to that of the state entanglement [44-46]. The symmetry-resolved operator entanglement in a specific symmetry sector q is defined through the renormalized Schmidt coefficients $\hat{\lambda}_{q,i_q} = \lambda_{q,i_q}/\sqrt{p_q}$ belonging to that sector via $S_{\text{op},q} \equiv -\sum_{i_q} \hat{\lambda}^2_{q,i_q} \log_2 \hat{\lambda}^2_{q,i_q}$, where i_q labels different Schmidt coefficients in the symmetry sector q and $p_q = \sum_{i_q} \lambda^2_{q,i_q}$ is the probability of having symmetry charge q in the half system such that $\sum_{i_q} \hat{\lambda}^2_{q,i_q} = 1$. In our SU(2)-symmetric case, different symmetry sectors are labeled by the half-system total spin S, and the probability of having total spin S in the half system is $p_S = (2S+1)\sum_{i_S} \lambda^2_{(S,i_S)}$ [note that in our notation for the SU(2) symmetry, each bond label (S, i_S) actually corespondents to $(2S+1)$ Schmidt coefficients with the same value λ_{S,i_S}]. Then the symmetry-resolved operator entanglement in the spin sector S is given by $S_{\text{op},S} = -(2S+1)\sum_{i_S} \hat{\lambda}^2_{(S,i_S)} \log_2 \hat{\lambda}^2_{(S,i_S)}$ with $\hat{\lambda}_{(S,i_S)} \equiv \lambda_{(S,i_S)}/\sqrt{p_S}$."

2. When the author introduces the model, they refer to Refs. [47,48] as papers that considered a similar model. These works were about superdiffusion and possible appearance of KPZ scaling in spin dynamics. Are there any connections between this work and superdiffusion/KPZ scaling? If so, this would be extremely interesting. But this is not discussed at all. Would the author be able to comment on this?

We thank the Referee for raising this interesting question. It has become a consensus that the superdiffusion, with dynamical exponent and asymptotic scaling profiles of dynamical structure factors belonging to the KPZ universality class, arises in one dimension due to a combination of non-Abelian symmetry and integrability [see, e.g., Ilievski *et al.*, PRX 11, 031023 (2021)]. In Refs. [47,48], a spin chain with fluctuating exchange couplings that is similar to the model considered in our manuscript was introduced to study the effect of breaking integrability on the fate of superdiffusive spin transport. The main result is that for a weak noisy exchange coupling that preserves the SU(2) symmetry, the diffusion constant was found to grow logarithmically in time, indicating superdiffusion in a weaker variety (see Ref. [47]), while in the strong noise limit considered in Ref. [48] the normal diffusion with a subleading correction $\propto 1/\sqrt{t}$ is restored. These results indicate that the spin transport properties highly depend on the noise (dissipation) strength. However, we numerically found that for the similar model, the operator entanglement exhibits the logarithmic growth behavior at late times both in the weak and strong dissipation regime. This suggests that the operator entanglement properties do not have too much connection with the spin transport properties.

3. In Ref. [38], an analytical undersanding for the logarithmic growth is reached by looking at the strong-dephasing limit, where the model maps to a classical stochastic model. Could something like this be done here in the SU(2) case? If so, could this provide insights into the observed phenomeon?

We thank the Referee for raising this point. As we clarified above, in the U(1)-symmetric case with dephasing considered in [Wellnitz *et al.*, PRL 129, 170401 (2022)], the analytical understanding is possible since the late-time operator entanglement in this case is fully captured by the classical Shannon entropy associated with the probabilities for the half system being in different U(1) sectors, which can be further related to the well-known classical symmetric simple exclusion process by the well-established perturbation theory in the strong dephasing limit. However, in our SU(2)-symmetric case, on the one hand, both the classical Shannon entropy and the symmetry-resolved operator entanglement in each symmetry sector contribute nontrivially to the late-time operator entanglement, on the other hand, even in the strong dissipation limit we need to identify the state ρ satisfying $\mathcal{L}_0[\rho] \equiv \sum_i (L_i \rho L_i^\dagger - \{L_i^\dagger L_i, \rho\}/2) = 0$ and the corresponding projector $\mathcal{P}$ onto these states to perform the perturbation analysis, which is also very challenging for our dissipation operators proportional to the dipole interaction between neighbor sites. These two facts make the analytical understanding of our results extremely hard. Nevertheless, it would be interesting to further develop new theoretical tools or construct much simpler models with the same symmetry to achieve an analytical understanding of the results presented in this work, which is left for future studies.

4. The author cites Refs. [41,42,43] and mentions the fact that symmetries can enhance entanglement of stationary states of systems with strong symmetries. But couldn't such a mechanism be invoqued here to explain the non-vanishing symmetry-resolved operator entanglement at long times? If not, why? It would be interesting if the author could elaborate on the connections or differences with these works.

We thank the Referee for raising this interesting point. Among these three references, the most relevant to our work is Refs. [42] and [43], in which both the authors studied the bipartite entanglement of the stationary states for unital quantum channels like Lindblad master equation with strong symmetries from the perspective of group theory and achieved very similar results. Especially, for the open quantum systems with non-Abelian continuous symmetries like SU(2) symmetry, both the authors found that the stationary states are highly entangled and the half-system operator entanglement scales logarithmically with the system size. The focus of these two works is actually very different from ours. In our work, we mainly focused on the dynamics of operator entanglement and considered an infinite chain, while these two works mainly investigated the static scaling of the entanglement with system size in the stationary states obtained at $t \to \infty$ for a finite chain. However, their results are consistent with ours. Especially, they showed that the operator entanglement at $t \to \infty$ diverges as the system size $L \to \infty$, while we predicted that the half-system operator entanglement for an infinite chain also diverges as $t \to \infty$. The nonvanishing symmetry-resolved operator entanglement at late times, especially in the limit $t \to \infty$, in our SU(2)-symmetric case, indeed can be understood from their results and is attributed to the nontrivial Hilbert space structure with dimension $d_J > 1$ for spin sector $J > 0$; see Eq. (20) in Ref. [42] or Eq. (23) in Ref. [43]. Without this property for SU(2) symmetry, only the classical correlations contribute to the operator entanglement at late times, like the U(1)-symmetric case considered in [Wellnitz *et al.*, PRL 129, 170401 (2022)]. We would like to note that although these two works found a similar logarithmic behavior of operator entanglement in system size for the stationary states at $t \to \infty$, it seems that the logarithmic behavior in the time domain do not have too much connection with the real space version as the prefactor η of the logarithmic behavior is different from each other. In this resubmission, we have added the discussion about the connections and differences between our work and Refs. [42,43] as follows:

"The nonvanishing contribution of symmetry-resolved operator entanglement at late times is compatible with the results shown in Refs. [42, 43], in which the authors studied the bipartite entanglement of the stationary states for unital quantum channels on a finite chain at $t \to \infty$ and found that the half-system operator entanglement scales logarithmically with the system size for the open quantum many-body systems with SU(2) symmetry. In this $t \to \infty$ limit, the nontrivial Hilbert space structure with dimension $2S + 1 > 1$ for spin sectors $S > 0$ leads to the divergent contributions from symmetry-resolved operator entanglement as the system size goes to infinity, otherwise the operator entanglement is fully captured by the classical correlations like the U(1)-symmetric case considered in Ref. [38]. We note that although the operator entanglement shows a logarithmic behavior both in system size in the $t \to \infty$ limit and in time in the limit of infinite system size, it seems that these two phenomena do not have too much connection with each other as the prefactor η in our work is different from those shown in Refs. [42, 43]."

Some weird sentences (mostly in the introduction) could be rephrased:

- 'three-rank tensor' $->$ 'rank three tensor'
- 'increasement' $->$ 'increase'
- 'Not only being useful in studying the closed systems'
- 'which is dubbed as the operator entanglement'
- 'it was recently studied that'
- 'we use MPO to represent the density matrix'

We thank the Referee for the suggestion. In this resubmission, we have rephrased these sentences as follows:

"In one dimension (1D), the quantum states can be faithfully represented by matrix-product states (MPSs) in terms of local rank-three tensors with bond dimension χ [22,23], for which the entanglement entropy is bounded by $\log_2 \chi$."

"Thus the linear increase of entanglement entropy will lead to an exponential growth of bond dimension, which is considered to be hard to simulate [24]."

"The growth of entanglement entropy also helps the understanding of open quantum many-body systems, which are ubiquitous in practical experiments due to the inevitable couplings to environments."

"The bipartition of MPDO through Schmidt decomposition further defines the entanglement entropy in operator space, which is known as the operator entanglement and characterizes the cost of an MPDO representation"

"However, it was recently reported that the operator entanglement in U(1)-symmetric open quantum many-body systems with dephasing increases logarithmically at long times [38]"

"To solve the Lindblad master equation, we describe the density matrix with MPDO."

Recommendation

Accept in alternative Journal (see Report)

We thank again the Referee for the review of our manuscript and the helpful comments/suggestions. Our manuscript has been carefully updated according to the above response. With all of the comments and suggestions being addressed, we wish the Referee would now agree the publication of our manuscript in SciPost Physics.