
Report of Referee 1

General remarks and recommendation

The preprint "Boundary operator expansion and extraordinary phase transition in the tricritical $O(N)$ model" by Xinyu Sun and Shao-Kai Jian (to be referred as SJ25 in the following) belongs to the series of contemporary papers written by young authors who try to discover and theoretically describe certain unusual phenomena that can happen in semi-infinite systems at bulk criticality, especially within their surface layer. They usually search for scenarios deviating from those described in the original classical paper [1] and [2] (not mentioned in SJ25). Efficient methods of the (Boundary) Conformal Field Theory and Conformal Bootstrap are involved, accompanied by extensive numerical computations. One of such theoretical methods has been developed quite recently in [3] and [4]. The technique of computing the layer susceptibility (LS) and hence deriving the two-point correlation function via Radon transformation has been successfully applied for the extraordinary transition (EOT) within the framework of the critical ϕ^4 theory in semi-infinite geometries. Shortly after that, Metlitski [5] showed the possibility of the new "extraordinary-log" boundary critical behavior in three-dimensional ($d = 3$) semi-infinite systems with continuous $O(N)$-symmetric order parameter. An essential feature of this scenario is the $1/(\ln x)^q$ behavior of the surface two-point function for large distances x at the bulk critical point. In SJ25, the authors modify and generalize the works sketched in the last two paragraphs in order to find out a possibly new boundary critical behavior at the bulk tricritical point. The significance of the manuscript under consideration is in that they seem to have succeeded. The idea of studying the semi-infinite tricritical systems is not new as it is seen from references 28, 29, 32, 34 and 54 in SJ25. However, the originality of SJ25 is provided by application of quite recent methods by Dey, Hansen and Shpot, and Metlitski to the problem. In my opinion, the article by Xinyu Sun and Shao-Kai Jian is original, up-to-date, scientifically sound and interesting, and thus deserves publication in SciPost Physics. It can be accepted with modifications implied by the following comments to be considered by the authors.

We thank the referee for the time and effort in reviewing our paper. We greatly appreciate the recognition of our work as original, up-to-date, scientifically sound, and interesting.

Comments and suggestions

1. In the Introduction, the authors refer to the tricritical O(N) universality class in Refs. [25–34], a rather chaotic and incomplete list with emphasis on polymers (corresponding just to the specific $N \to 0$ limit). In the present statistical-physics context, a standard classical reference to the subject is [6], which is not mentioned in the list. This reference provides many more different physical applications beyond that mentioned in SJ25. Moreover, consulting this reference would help authors in writing their Conclusion section in this context. It would also be appropriate to include the reference [54] into this list. It is just an extended version of Ref. [32]. The same could be done with [56].

We thank the referee for the comment. To make the reference list clearer, we have moved the references on polymers to the conclusion section, where we specifically discuss the

application of the tricritcal theory to the θ point of the polymers. Also, we have included Ref. [6] (Theory of Tricritical Points, in Phase Transitions and Critical Phenomena), as well as Refs. [54] and [56], in the list in the revised manuscript.

2. There are numerous situations in statistical physics (disordered $O(N)$ models, systems with cubic anisotropy) where certain critical numbers N_c of the number of components of the order parameter field appear, like in SJ25. Besides, using the term "critical flavor" for N_c, borrowed from quite a different field of high energy physics looks strange here. I would suggest not to use it.

We thank the referee for the advice. Following the suggestion, we have replaced all instances of "critical flavor" with "critical number of components" in the revised manuscript.

3. On p. 5, specify that "we establish the extraordinary transition for any N..." in $d = 3$.

We have specified $d = 3$ in the revised manuscript.

4. In (18), as well in (91) and (92), the BOE coefficient c_4^L appears with the coefficient $N - 1$, so that $N = 1$ is the special case when $c_4^L(N)$ vanishes. It is the only one of BOE coefficients that has this property. Do you understand the physical reason for this, and can you somehow comment on it? A related suggestion: indicate that the sum over Δ in (91) runs starting with $\Delta = 4$.

We thank the referee for the comment. Analytically, the prefactor $N - 1$ arises from the Feynman diagram c_{24} shown in Fig. 3 (b), where the loop is formed by propagators of the transverse modes, yielding the $N - 1$ factor. This gives rise to Eq. (81), which includes the higher-order term $H''(\zeta)$. The leading contribution to $H''(\zeta)$ is of order ζ^4. In contrast, the term $H'(\zeta)$ originates from the Feynman diagram c_{23}, also shown in Fig. 3 (b), where the loop consists of propagators of the longitudinal modes. Its leading contribution appears at order ζ^6. Hence, c_4^L arises exclusively from the transverse mode, explaining the factor of $N - 1$. We have added this discussion to the revised manuscript. Besides, we have also indicated the sum over Δ in Eq. (92) starting with $\Delta = 4$.

5. If you wish to use the shorthand NLSM in Sec. 5.2 and throughout the paper, introduce it on p. 6 where the term "nonlinear sigma model" appears for the first time.

In the revised manuscript, we introduced the abbreviation "NLSM", when the term "nonlinear sigma model" first appeared.

6. P. 7: saying that "...the expectation of the surface order does not vanish", specify the value of this limit. At the end of the same paragraph, please explain why do you refer to [22] here.

We have specified the value of this limit that $\langle n \rangle_r \to 1$. We have removed the reference to Cuomo and Zhang from this subsection and instead incorporated the discussion and the reference into the introduction to better motivate the context and significance of our work.

7. I would encourage the authors to write down "the expansion of $G^T(\xi, z_1, z_2)$ as a series of ξ^n with known prefactors" if you have derived it.

The expansion of $G^{L,T}(\xi, z_1, z_2)$ as a series of ξ^n can be obtained using Eq. (23) in the main text, which we also present below:

$$G_{\rm con}(r-r', z, z') = (4zz')^{-\Delta_\phi} \sum_{\Delta>0} c_\Delta \sigma_\Delta \mathcal{G}_{\rm boe}(\Delta, \xi), \tag{R1}$$

$$\sigma_\Delta = 4^{-\Delta+\frac{d-1}{2}} \pi^{-\frac{d-1}{2}} \Gamma(\Delta)/\Gamma(\Delta - \frac{d-1}{2}), \quad \xi = \frac{(r-r')^2+(z-z')^2}{4zz'}, \tag{R2}$$

$$\mathcal{G}_{\rm boe}(\Delta, \xi) = \xi^{-\Delta}\, {}_2F_1\left(\Delta, \Delta+1-\frac{d}{2}, 2(\Delta+1-\frac{d}{2}); -\xi^{-1}\right). \tag{R3}$$

By substituting the BOE coefficient $c_\Delta^{L,T}$, we obtain the series expansion of $G^{L,T}(\xi, z_1, z_2)$ for the longitudinal and transverse component, respectively. Since we have derived the analytical results of $c_\Delta^{L,T}$, we could attempt to explicitly sum over Δ and simplify the result. However, to the best of our knowledge, we were unable to find a closed-form expression for the summation.

Formatting and level of grammar

The overall formatting of the manuscript is good, in particular I welcome a short presentation of the main results in a separate section following the Introduction. Some further suggestions concerning the formatting are:

1. Move the text on p. 3 starting from "The methods developed in this study lay a foundation..." until the end of the paragraph to the end of Conclusions. Anyway, the final sentence in Sec. 6 is almost a direct copy of this fragment.

We thank the referee for the advice. We have moved the text from p. 3 to the end of the conclusions section and have rephrased the sentence accordingly.

2. On p. 15, the sentence including (70) is a "copy-paste" of the last sentence in Appendix B2. This repetition is not necessary. It is enough to show the final result (70) in the main text, and in B2 — just refer to it. One has to check whether there are no other similar repetitions in the text.

We have removed the repetition in Appendix B2 and carefully checked the text for any other similar redundancies.

Apparently, the paper has been written by a "big boss" (BB) and a young "hard worker" (HW). The manuscript is a compilation of portions written by BB and HW. The parts by BB and HW are different in style and of different grammar level. There are too many grammatical issues in the part of HW. On the other hand, the style of BB is a kind of spoken-language-in-a-hurry and tends to be inaccurate.

— For example: in the second paragraph on p. 21 we learn that "the bulk action will flow to the normal boundary condition". Is it possible for the action to flow to the boundary condition?

— And: "The normal boundary condition is actually equivalent to the extraordinary transition". Can really a boundary condition be equivalent to any transition?

— Or: "the IR description of the extraordinary": extraordinary "what"?

No double check and final reading of the submitted version has been done by the authors. Apparently, HW was already too tired for this, and BB did not find time for that. Apart from grammatical errors and inconsistencies in mathematical formulations — to be corrected by the authors, I have noticed several essential misprints:

We appreciate the referee's thorough review and apologize for the oversights in the submitted version. We have reviewed the manuscript carefully, corrected grammatical errors and inconsistencies in mathematical formulations, and addressed the essential misprints noted by the referee.

1. P. 19: While this can be done explicitly for the critical $O(N)$ model with $N = 2$

→ While this can be done explicitly for the critical $O(N)$ model with $n = 2$

2. P. 32, the first line below (144): $b = \frac{n-2}{2\pi} \to b = \frac{N-2}{2\pi}$

3. P. 33, (146): $\frac{5}{2}$ in Bessel functions should be indices, as in (43)

4. P. 33, (151): the argument in the second ${}_5\tilde{F}_4$ has to be 1 instead of y

We thank the referee for pointing out these typos. We have corrected them according to comments 1-4.

5. P. 33 and 34: Eq. (3.13) from [3] should be changed to (3.15) – (3.16): for some reason, the equations' enumeration in the journal article [3] and its ArXiv version are different.

We thank the referee for pointing out this difference, and have corrected it to (3.15) – (3.16) in the revised manuscript.

6. Check the consistency of notations: For example, in equations (138) and (140), as well as in (141) and (143), the same function is denoted by H and H_d.

We have changed all H_d, H'_d and H''_d to H, H' and H'' in the revised manuscript.

7. The term "Callan-Symanczyk equation" is copy-pasted from the paper by Metlitski ([5], p.10) without checking. First: the correct spelling is Symanzik. Second: the Callan-Symanzik equations are the ones used in the massive field theory. In the context of SJ25 it is appropriate to use the term "the renormalization group equation" instead.

We have changed the term to "the renormalization group equation" in the revised manuscript, as suggested.