Reply to Referee 1

Thank you very much for submitting this paper to SciPost. I gave my best to faithfully assess the quality of this paper and hopefully help to improve it with my comments.

We greatly appreciate the effort of the referee in assessing our work. We acknowledge that the mathematical part is dense and not the easiest to digest. Nevertheless, we strongly believe that their comments have helped us improve the manuscript, and we are sincerely grateful to them.

This paper studies the fidelity decay in random quantum circuits with focus on swap operations. The considered model interleaves layers of 2-qubit gates with arbitrary permutations. The authors analyze the effect of faulty 2-qubit gates and faulty permutations, implemented through combinations of malfunctioning swap gates. For the ease of the analysis, this model is surrogated with a solvable model in which the permutations are substituted with $\Pi \to R\Pi R$*, for* R *being global unitaries sampled from the Haar-random distribution.*
The paper is well explained and written, and it follows a logical line of thought of creating an easily manageable model to deal with a complicated real scenario. The results are as one could expect from error accumulation in quantum computing: the errors will decay exponentially with the quantum volumen, i. e., LT*, implying that errors will take over the computation unless the error rate decays at least at that pace*.

We thank the referee for their careful reading of our manuscript and for the positive evaluation of our work. Indeed, the qualitative behaviour—namely, the exponential decay of fidelity—is not unexpected. However, we would like to emphasize that our contribution lies in providing a quantitative description. Previous work has so far considered extremely small errors and expanded the fidelity to the first nontrivial order. In contrast, we performed the full calculation of fidelity, providing a constructive proof of the conjectured exponential decay.

It is important to clarify that the notion of *quantum volume* usually refers to the dimension of the circuit's Hilbert pace 2^L, where L is the maximal number of qubits such that a square quantum circuit ($T = L$) is deemed satisfactory—i.e., "quantum"—under a specific passing criterion. Therefore, it is important to distinguish this from the dependence that appears in fidelity. Indeed, the "volume-like" decay LT, to which the referee refers, arises only in the case of full connectivity and suggests that in this scenario regarding error accumulation, the circuit's length (L) and depth (T) are on equal footing. In contrast, our work goes further: we show that when connectivity is limited, more gates are inevitably required, leading to a decay governed by L^2T for 1 dimensional architecture and $L^{1+\frac{1}{d}}T$ for general d-dimensional architecture. (see Eq. (11) and Eq. (13)).

I have two main concerns. First, it would be very interesting to have an interpretations on why the solvable model yields accurate results for the original model. From a information-theoretic point of view, one could think that the random gates R *scramble the 2-qubit gates and erase all information. Simple 2-qubit gates plus permutations have the same asymptotic effect, but the numerical results show agreement even for low* T*. In this sense, I miss some interpretation on why the solvable model works so well.*

We thank the referee for raising this important concern. Indeed, from an information-theoretic perspective, the two models are quite different: their capacity to scramble information, their patterns of entanglement growth, and their feasibility for implementation on real hardware all differ significantly due to the presence of all-to-all Haar-random unitaries in the solvable model. However, since our calculations are primarily focused on fidelity, which requires alignment only on the level of 2 designs, we were able to modify our model without altering the results.

As discussed in the manuscript, the computation of fidelity involves two copies of the unitaries (and their conjugates). The strong similarity between the original and solvable models stems from the fact that the original layer $\Pi_\tau V_\tau$ belongs to an ensemble that approximates a 2-design, and it converges exponentially toward a 2-design as the circuit depth increases. Consequently, augmenting the circuit with unitaries drawn from the Haar measure (which by definition is also a 2-design) does not alter the ensemble statistics at the level of second moments. In summary, the referee's intuition is absolutely correct: the solvable model *works so well* precisely because we are analyzing the circuit's behaviour at the level of second moments.

Finally, we agree that the average over only large random unities R would scramble the action of local unitaries and permutations. However, since we aim to calculate the average over those gates, they are already effectively scrambled, and the additional average does not substantially change the result.

My second concern is about the utility and impact of this research. Precisely because the results are not surprising, I do not see any special novelty in this manuscript, which is not captured by other noise models, such as Pauli depolarization. I am no expert in the field and I do not have a complete overview of the literature. Hence, I would appreciate a section in which the authors clarify what is the relationship of this work with the existing literature in the field.

On the weaknesses side, I miss some clear statement of the motivation and prospects of this research. However, this is a personal opinion, and implementing changes in this direction should be taken only as a suggestion.

We thank the referee for pointing out a possible weakness in our work. As we signalled above the main novelty of our results lies in the quantitative description of fidelity decay, in quantum devices affected by structured noise. We presented explicit dependence on such parameters as the strength of possibly different noises affecting 2-qubit gates and permutations, or quantum computer architecture. To better motivate the explicit noise model chosen for our calculations, we extended the last paragraph in Section 2.1:

"Since the two-qubit gates are already random, noise must be modelled as a random deviation from the uniform sampling defined by the Haar measure. To this end, to preserve the symmetry with which the gates were sampled we consider that each random unitary $u_{r,r'}$ is independently perturbed by unstructured noise: $\tilde{u}_{r,r'} = e^{i\alpha h_{r,r'}} u_{r,r'}$, where $h_{r,r'}$ is drawn from the Gaussian Unitary Ensemble (GUE), and $\alpha \geq 0$ controls the noise strength. Notably, the ability to model noise independently in both the permutations and the two-qubit gates provides significant flexibility and control in our framework."

We have also added the following paragraph in the introduction that emphasizes the motivation and usefulness of our work.

"In the context of quantum chaos and dynamic phase transitions, quantum fidelity is termed Loschmidt echo[3, 2], measuring the extent to which a complex system is recovered after applying an imperfect (perturbed) time-reversal. In the framework of time-independent Hamiltonians, the behaviour of the Loschmidt echo is well understood for single-particle quantum systems whose dynamics are fully chaotic in the classical limit: it typically exhibits an initial parabolic decay, followed by an exponential one, and eventually saturates [2]. This pattern has also been observed in many-body systems [4], and similar behaviour is expected in systems governed by time-dependent Hamiltonians, such as quantum circuits [1]. While a quantitative understanding is valuable in its own right, it becomes particularly pertinent in light of the technological relevance of quantum circuits."

Moreover, although general exponential-like decay is not surprising, the explicit dependence on quantum architecture was, to our knowledge, not presented before.

Other minor issues:

- *In the title of Section 3 there is a typo: Sovable*

We apologize for such a big typo in the section title, and we thank the referee for pointing it out.

- *In Fig. 2, the choices of α seem arbitrary to me, I do not see any particular motivation for those values.*

As shown in Fig. 2(b), we explore the regime of small α more thoroughly, motivated by the fact that this is the relevant regime in the context of quantum computing, where noise must be minimized as much as possible. The choice of α in Fig. 2(a) is, in fact, somewhat arbitrary and was made primarily for aesthetic convenience.

- *I have my doubts that Quantum Volume is a well-established measure for complexity.*

We fully agree with the referee. The notion of *gate* complexity refers to the minimal length of a circuit that reproduces (within some tolerance) a desired unitary operation. It typically assumes ideal, faultless gates and is generally a very challenging problem. In contrast, Quantum Volume is a figure of merit for physical quantum *computers* that, by definition, accounts for the presence of noise. It quantifies the size of the largest square-shaped circuit that can be reliably executed.

References

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[2] Arseni Goussev et al. "Loschmidt Echo". In: *Scholarpedia* 7.8 (Aug. 2012), p. 11687. ISSN: 1941-6016. DOI: 10.4249/scholarpedia.11687. (Visited on 07/30/2024).

[3] Markus Heyl. "Dynamical Quantum Phase Transitions: A Review". In: *Reports on Progress in Physics* 81.5 (Apr. 2018), p. 054001. ISSN: 0034-4885. DOI: 10.1088/1361-6633/aaaf9a. (Visited on 07/30/2024).

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