I. REPLY TO REFEREE 2

We thank you for reviewing our manuscript and find our work easily meets expectations and criteria for *SciPost Physics* among top 50%. Below, we address your comments.

1. **Referee 2:** *The term "topology-induced symmetry breaking" (TISB) is compelling but perhaps deserves a clearer definition upfront, with a contrast to spontaneous symmetry breaking in the introduction.*

   Response: At the end of Sec. 1.2, we have modified our phrases with a clear definition of TISB and a contrast to pontaneous symmetry breaking.

2. **Referee 2:** *The "exotic phase" with **ferromagnetic interchain couplings** but preserved AFM order is a particularly intriguing prediction. The analysis of the soft mode and stability threshold is convincing. Could there be **experimental observables** that directly detect the spectral shift or suppressed chirality, beyond the proposed time-resolved microscopy?*

   Response: Beyond time-resolved imaging, polarization-resolved Brillouin light scattering (BLS) [e.g., see *Physics Reports 348.6 (2001): 441-489*] could directly reveal the α–β frequency splitting and the absence of circular polarization: by analyzing incident and scattered polarization states, it resolves two peaks at the α and β frequencies while the circular channel vanishes, confirming suppressed chirality.

3. **Referee 2:** *A brief discussion of **fermionic systems**, such as electrons in a Möbius ring, is mentioned in the outlook but not developed. A qualitative contrast with the magnon case would enrich the discussion.*

   Response: Fermionic systems differ fundamentally in their wavefunction structure, boundary conditions, and diagonalization procedures, so it would be premature to draw definitive conclusions without a dedicated analysis. Accordingly, we have revised the "Conclusion and Outlook" section as follows:

   "For tight-binding electrons on a Möbius strip, the wavefunction is a two-component spinor $\Phi(\boldsymbol{r}) = [\phi_\uparrow(\boldsymbol{r}), \phi_\downarrow(\boldsymbol{r})]^{\mathrm{T}}$, and the TBC in Sec. 1.2 becomes: $\hat{T}_L\Phi(x, y) = -\Phi(x + L, -y)$.

While the Hamiltonian can formally parallel our magnon model, the fermionic statistics also brings a fundamental distinction in the diagonalization [which does not involve σ_z as that in obtaining Eq. 15]. Consequently, the precise impact of TBCs on electronic bands also merits a dedicated, detailed study."

4. **Referee 2:** *Minor comment: In several places, "eigenmodes of excitation" could simply be "eigenmodes" or "excitations" for conciseness.*

   Response: We use the phrase "eigenmodes of excitation" intentionally to stress that it is the modal structure of the excitations—not merely their energies or other properties—that is disrupted. Furthermore, in Sec. 1.2 we write:

   > ... fundamentally disrupting the eigenmodes of excitations of all kinds accompanied by chirality.

   Omitting "of excitations" risks obscuring that we mean the excited eigenmodes rather than the magnetic ground state.