Dear Editor,

In spite of the initial rejection, we are grateful to the referee for his/her observations and, in particular, for the two references, the second of which we did not know and to which we will refer in the revised version of this manuscript. We are also grateful for giving us the opportunity to appeal and explain the novelty and technical soundness of our results. We are confident that the explanation that follows will be convincing enough to review the rejection.

The basic (and most important) question the referee asks is "what are the new results of this paper?"

Our paper has a long introduction in which we review known results which, by definition, are not new. We believe we have referred to most of the relevant papers on this topic, except for the second one mentioned by the referee. (The first reference mentioned by the referee was known to us and it is not relevant, as we are going to explain. In particular, it gives a different Smarr formula. Ours coincides with the one in [34] and [35]).

Yes, the Komar charge we find at the end was known. However, some of the intermediate results were not known, because, being interested in the Komar charge, most researchers have made the derivation on-shell, with vanishing torsion, with the notable exception of Prabhu in reference [22] who, however, uses a complicated formalism and notation. Our derivation uses a formalism which is more standard in the physics literature. Working off-shell one obtains the off-shell closed Noether current (27) (closed for any vector field ξ, and not just for Killing vectors!) and the Noether–Wald charge (29) which coincides with Prabhu's. The off-shell closed Noether current does not coincide with Prabhu's because he uses a different definition which is only closed on-shell. This definition is often used in the literature because it is simpler to compute, but it unnecessarily obscures the procedure.

Yes, in the second reference indicated by the referee our definition (46) for the NUT charge was already proposed in a rather *ad hoc* way, not derived from an action. However, neither in that reference nor in any other reference it was observed that the very definition of mass is modified by the additional term we find in the Komar charge in the same way in which a θ term modifies the definition of electric charge in electrodynamics. In that context it is known as "Witten effect" (Re. [42]). The modification is such that a spacetime whose mass was zero using the standard Komar charge but with non-vanishing NUT charge, would acquire a non-vanishing mass according to the new definition of mass that follows from the new Komar charge. This is a "gravitational Witten effect" and this is our main result, which we believe is new.

We also discuss whther and how this effect modifies the Smarr formula. Since the effect had not been noticed before, its effect on the Smarr formula had not been considered before, either.

Summarizing: the technical derivation of the Komar charge is not entirely new, because it was made in 2+, albeit in a different, more obscure, language, and the study of the physical consequences of the presence of a new term (the gravitational Witten effect) is entirely new.

We believe that these new results should grant the consideration of our paper for publication.

Concerning the technical observations at end of the referee's report concerning the Noether and Komar charges and their off-shell/on-shell closure, we would like to make a few comments before writing the proof requested by the referee:

1. In the Iyer-Wald '94 paper the Noether-Wald charge is only computed for the case of pure GR (GR in vacuum), in spite of the (wrong) initial discussion over matter fields. (Notice that the first law that they write does not contain any work terms associated to charges or potentials.) Given that in this theory (and, basically, only in this theory) the Noether-Wald charge coincides with the on-shell-closed Komar charge, one can not draw general conclusions from it.

2. There are better general references on the topic of symmetries and conserved charges like, for instance, Barnich and Brandt's or Compère and Fiorucci's. The definition of the Noether current in the case of local symmetries can differ by the terms involving the equations of motion but only when you include them is the current off-shell closed and the existence of a Noether charge follows from Poincaré's Lemma.

3. It is a well-known fact that in theories invariant local symmetries (which we will call gauge theories for short, independently of whether we are talking about Yang–Mills-type gauge symmetries, g.c.t.s or local supersymmetry transformations), this invariance has several consequences

   (a) The Noether current $(d-1)$-form $\mathbf{J}[\xi]$ athat we define[1] is closed off-shell

   $$d\mathbf{J}[\xi] = 0\,, \tag{1}$$

   and it cannot be used to define conserved quantities. We will show how it has to be defined in order to have this property.

   Notice that, for global symmetries, this off-shell definition of the Noether current is simply not possible. One has to consider a Noether current which is only on-shell closed.

   (b) This closure implies the existence of a Noether[2] $(d-2)$-form charge $\mathbf{Q}[\xi]$ related to $\mathbf{J}[\xi]$ by

   $$\mathbf{J}[\xi] = d\mathbf{Q}[\xi]\,. \tag{2}$$

[1] In this discussion, ξ stands for any kind of gauge parameter.
[2] We call this charge Noether-Wald charge when we are dealing with g.c.t.s.

If one is given $\mathbf{Q}[\xi]$ and proves this relation, the closure of $\mathbf{J}[\xi]$ follows automatically.

If one uses the other definition of Noether current which is only closed on-shell, the above relation is only true on-shell.

(c) The existence of relations between off-shell l.h.s.s of the equations of motion known as Noether identities which imply that not all of them are independent and that the theory has less degrees of freedom. The use of these off-shell identities is essential in the proof of the off-shell identity Eq. (1). The referee mentions the failure of dE' to vanish off-shell, but does not take into account the existence of these off-shell relations between the equations of motion (and their derivatives).

4. The "trivial" conservation of the Noether current in these theories makes it necessary to find a different definition of conserved quantities that does not involve the integral of the Noether current, because, as the referee correctly indicates, it is associated to a trivial conservation law. This problem cannot be fixed by using the other definition of Noetehr current which is only closed on-shell because it satisfies the on-shell version of Eq. (2). This is why in gauge theories conserved quantities are defined as integrals of $(d-2)$-forms over the boundary.

5. Constructing the $(d-2)$-forms that define the conserved quantities for general configurations is a complicated problem (see the ADM paper, or Abbott and Deser's or the above reference by Barnich and Brandt for the case of the energy and momentum in GR). However, when the configurations have enough symmetry and are solutions of the equations of motion there are procedures that allow one to find $(d-2)$-form charges whose integrals give the conserved charges. In pure GR, this is the Komar charge associated to a given Killing vector, which is closed on-shell and for Killing vectors (*i.e.* for symmetric configurations and in GR coupled to matter it is the generalized Komar charge[3]).

6. As mentioned above, the Komar charge of pure GR coincides with the Noether-Wald charge. However the Noether-Wald charge of a generic theory is not closed on-shell. This is a well-known problem, but there are ways to construct on-sell-closed generalized Komar charges combining the Noether-Wald charge with other term [4]

Before embarking in a long explicit calculation, we would like to indicate that the definition of the off-shell-conserved Noether current in our manuscript, (27) follows

[3]Spacetimes which are not static or stationary but are asymptotically flat have a well-defined ADM mass that cannot be calculated through the generalize Komar charge because they meet the requirement of being solutions but not that of being symmetric.

[4]See, for instance, JHEP **09** (2025) 068. A general construction of the Noether current involving the equations of motion and a proof of its off-shell closure/conservation can be found in it as well as abundant literature on on-shell closed generalized Komar charges.

easily from the preceding equations: using the Noether identities (26a) and (26b) which are explicitly demonstrated in footnote 9, in (22), we see that the first two terms disappear and we are left with the total derivative

$$\delta_\xi S = \int d\left[\mathbf{\Theta}(e,\omega,\delta_\xi\omega) + \mathbf{E}_a\xi^a + \mathbf{E}_{ab}P_\xi{}^{ab}\right] . \tag{3}$$

On the other hand, (18) provides an alternative expression for $\delta_\xi S$. Equating these two equations, we find the off-shell identity

$$\int d\left[\mathbf{\Theta}(e,\omega,\delta_\xi\omega) + \mathbf{E}_a\xi^a + \mathbf{E}_{ab}P_\xi{}^{ab} + \imath_\xi \mathbf{L}\right] = 0 . \tag{4}$$

Since this result is valid for any integration region, we conclude that

$$\mathbf{J}[\xi] \equiv \mathbf{\Theta}(e,\omega,\delta_\xi\omega) + \mathbf{E}_a\xi^a + \mathbf{E}_{ab}P_\xi{}^{ab} + \imath_\xi \mathbf{L} , \tag{5}$$

is off-shell closed.

We must insist in the fact that this is what should be expected in any gauge theory.

Now, we are going to prove directly that the exterior derivative of the Noether–Wald charge given in Eq. (29) of the manuscript, ignoring the factors of $16\pi G_N^{(4)}$

$$\mathbf{Q}[\xi] = -X_{abcd}e^a \wedge e^b P_\xi{}^{cd} , \tag{6}$$

gives the Noether current as defined above, *i.e.* we are going to prove directly Eq. (30) of the manuscript or Eq. (2) here. The consequence will be that $d\mathbf{J}[\xi] = d^2\mathbf{Q}[\xi] = 0$ off-shell, wjich is what the referee wnats us to prove. Since the tensor X_{abcd} is covariantly constant under a Lorentz connection because it is constant and Lorentz-invariant

$$d\mathbf{Q}[\xi] = -2X_{abcd}\mathcal{D}e^a \wedge e^b P_\xi{}^{cd} - X_{abcd}e^a \wedge e^b \wedge \mathcal{D}P_\xi{}^{cd} . \tag{7}$$

Now, using the Cartan structure equation (23a)

$$\mathcal{D}e^a + T^a = 0 , \tag{8}$$

and Eq. (62)

$$\delta_\xi\omega^{ab} = -\left(\imath_\xi R^{ab} + \mathcal{D}P_\xi{}^{ab}\right) , \tag{9}$$

and recognizing the equations of motion of the Vierbein (16a),

$$\mathbf{E}_a = -2X_{abcd}e^b \wedge R^{cd} , \tag{10}$$

the Lorentz connection (16b),

$$\mathbf{E}_{ab} = 2X_{abcd}T^c \wedge e^d , \tag{11}$$

the presymplectic potential (17),

$$\mathbf{\Theta}(e,\omega,\delta\omega) = X_{abcd}e^a \wedge e^b \wedge \delta\omega^{cd}. \tag{12}$$

and the Lagrangian (13)

$$S[e,\omega] = \frac{1}{16\pi G_N^{(4)}} \int X_{abcd}e^a \wedge e^b \wedge R^{cd}\,, \tag{13}$$

we get

$$\begin{aligned} d\mathbf{Q}[\xi] &= 2X_{abcd}T^a \wedge e^b P_{\xi}{}^{cd} + X_{abcd}e^a \wedge e^b \wedge \imath_\xi R^{cd} + X_{abcd}e^a \wedge e^b \wedge \delta_\xi\omega^{cd} \\ &= \mathbf{E}_{cd}P_{\xi}{}^{cd} + \imath_\xi \left(X_{abcd}e^a \wedge e^b \wedge R^{cd}\right) - \imath_\xi \left(X_{abcd}e^a \wedge e^b\right) \wedge R^{cd} + \mathbf{\Theta}(e,\omega,\delta_\xi\omega) \\ &= \mathbf{E}_{cd}P_{\xi}{}^{cd} + \imath_\xi\mathbf{L} - \xi^a 2X_{abcd}e^b \wedge R^{cd} + \mathbf{\Theta}(e,\omega,\delta_\xi\omega) \\ &= \mathbf{E}_{cd}P_{\xi}{}^{cd} + \imath_\xi\mathbf{L} + \xi^a\mathbf{E}_a + \mathbf{\Theta}(e,\omega,\delta_\xi\omega)\,, \end{aligned} \tag{14}$$

quod erat demonstrandum.

We should stress that this derivation, in this formalism, has not been published before in the literature. We notice that, in the definition of conserved charges for spacetimes with no symmetries, these intermediate results are needed, since one only demands that the vector field ξ should tend asymptotically to a Killing vector of spatial infinity.

We hope that this detailed explanations will grant the acceptation of our manuscript for publication in SciPost. Should our manuscript be accepted we would improve the presentation of the derivation of the off-shell closed Noether current and Noether–Wald charge, apart from including the second reference mentioned by the referee.

Yours,

Tomás Ortín (as corresponding author)