Dear Editor, Dear Referees,

we would like to thank the referees for their time and a thorough analysis of our manuscript. We are delighted about the positive comments made by the referees and are grateful for the many helpful comments. Below we directly address all of the comments made by Referee 1. We incorporated all of the suggestions and comments made by the referees, which we summarized at the end of this document. We believe that our manuscript is now ready for publication in SciPost.

Sincerely,
Matjaž Kebrič, on behalf of all authors

Report 1

The authors study a model of spinless hard-core bosons on a 1D lattice with a p-wave pairing term and coupled to a dynamical Z2 gauge field. By Jordan-Wigner transformation, this model is equivalent to the same model with hard-core bosons replaced by fermions, that was studied previously in Ref. 28. Using mean-field theory and DMRG simulations, the authors vary model parameters such as filling, strength of pairing term, electric string tension and map out the ground-state phase diagram, largely confirming the results obtained in Ref. 28. In addition to the entanglement entropy, the authors also compute string-length histograms/net electric field polarization and boson Green's function to diagnose (de)confinement.

The paper is clear, detailed, and well written. The work presents interesting results that enrich our understanding of (discrete) lattice gauge theories (LGTs) with matter. This is a topic of enduring interest that has picked up speed in recent years, largely motivated by new experimental prospects for the realization of LGTs in synthetic matter (e.g. cold atoms and quantum computers).

Our reply

We would like to thank the referee for their time, and are delighted by the positive assessment of our work.

I would like to offer the following comments/questions, in order of decreasing importance:

1. My main concern regarding possible publication in SciPost Physics is the novelty/importance of the work. First, a large fraction of the manuscript is devoted to confirming results previously obtained in Ref. 28 (if not quantitatively, at least qualitatively). I believe most (if not all) the phases appearing in the phase diagrams presented here had already been found, and in the same model, by Ref. 28. Second, mean-field methods have a long history in LGT (see, e.g., Drell, Quinn, Svetitsky, Weinstein, PRD 19, 619 (1979); Boyanovsky, Deza, Masperi, PRD 22, 3034 (1980); Horn, Weinstein, PRD 25, 3331 (1982); and probably several others), but the authors present their mean-field approach for Z2 gauge theory as an entirely new idea. Regarding both points, which contributions would the authors say constitute important, conceptually new results? The authors should state this very clearly and also explain the relation to previous work more carefully (with appropriate citations as necessary).

Our reply

Our results indeed agree with the phase diagram obtained by Borla et. al in Ref. 28, however we believe we provide a much more deeper understanding of the system on the microscopic level. For example Borla et. al focuses on gauging the Kitaev chain, and all of the numerical calculations is focused on the entanglement entropy calculations as a function of K, which is the parameter of the gentle gauging of the Kitaev chain, with $K \to 0$ reducing the model to the pure Kitaev chain and $K \to \infty$ is the limit of the fully gauged Kitaev chain. From this we also see that their numeric focuses entirely on the regime where $|\lambda| = 1$.

Our research is mainly motivated by the quantum simulation experiments, both analog and digital. We are thus interested in how different phases of the rich phase diagrams of the $\mathbb{Z}_2$ LGT can directly be probed in the experiment. As a result we studied the string-length histograms which provide a direct geometric probe of confinement, which can be probed in quantum simulation experiments. The work by Borla et. al in Ref. 28, provides a thorough analysis of the SPT phases of the system and the effect of gently gauging the Kitaev chain, and we refer to their results throughout our paper. However, we believe that our analysis provides novel insights from the macroscopic perspective and is directly relevant to the quantum simulation community. In particular we study the effect of doping and the understanding of the phase diagram on the microscopic level, Hence, we present our results as a function of the chain filling $n = N/L$, where N is the average parton number in the chain length L. We dive into a detail analysis of confinement by considering the $\mathbb{Z}_2$ Greens's function and the string-length histograms. With that we directly probe the confinement across the wide parameter regime as a function of filling, and the SC term λ.

The second part of our paper, which we consider as equally important is the derivation and study of the mean-field theory. Mean-field methods have been indeed used before in the study of LGTs, and thus the idea of using a mean-field approach is not new. We do not claim in our work that we are the first to develop a mean-field theory, and indeed the citations provided by the referee show that a lot of work has been done on the two and three dimensional systems. However, we believe that our study provides an important new results, as we develop a mean-field theory for a one-dimensional $\mathbb{Z}_2$ LGT, which to the best of our knowledge was not done before for this particular system. In such a way, our work provides new insights into how the slave-parton mean-field approach can be successful in low dimensions, which goes against the conventional wisdom where mean-field methods usually fail at low dimensions. Moreover our mean-field derivation also shows how we can obtain a transverse field Ising model in the limit when $\lambda = -t$. It thus provides a new intuitive picture of the $\mathbb{Z}_2$ LGT.

We agree with the referee that references to previous mean-field results in higher dimensions would provide an additional information on where our mean-field approach stands in comparison to the previous work. As a results we included the following sentence in the beginning of Sec. IV., with the additional references to the mean-field approaches in higher dimensions:
"We note that variational mean-field approach has been successfully used to study transitions in LGTs in higher dimensions [51–53].", where [51-53] are now:
[51] S. D. Drell, H. R. Quinn, B. Svetitsky, and M. Weinstein, Quantum electrodynamics on a lattice: A Hamiltonian variational approach to the physics of the weak-coupling region, Physical Review D 19, 619 (1979).
[52] D. Boyanovsky, R. Deza, and L. Masperi, Variational method for the Z(2)gauge model, Physical Review D 22, 3034 (1980).
[53] D. Horn and M. Weinstein, Gauge-invariant variational methods for Hamiltonian lattice gauge theories, Physical Review D 25, 3331 (1982).

2. In the mean-field Hamiltonian (8), the authors drop the electric field term $\langle \tau_{xi,j} \rangle$

on account that it is a constant energy offset. I am concerned about this because this constant can take different values in different variational states, and thus potentially affect the phase diagram (which should reflect which state has the absolute lowest energy).

Our reply

We drop this term since we decouple the gauge and matter degrees of freedom. In Hamiltonian in Eq. 8, this becomes a constant term since it does not couple to any hard-core boson operators. In such sense it does not change the physics of the matter Hamiltonian and we believe it can be dropped. We did not drop this term in our effective Hamiltonian for the gauge fields $\hat{\tau}$, which would indeed be unjustified.

3. The authors argue that the antiferroelectric state $<\tau_{xj,j+1}>\sim(-1)^j$

is stable upon turning on a nonzero h term, while the ferroelectric state $<\tau_{xj,j+1}>\sim$ const. is not. This confuses me for the following reason. For h=0, the model (1) actually has a global Z2 symmetry under reversal of all the τ^x electric fields, which is generated by a global Wilson loop $W=\prod_j \tau_{zj,j+1}$ around the entire lattice. (Note that this symmetry also preserves the Gauss' law constraint.) First, the authors should explicitly mention this global symmetry of the model after they introduce it, especially since they often constrast the h=0 and $h\neq0$ cases throughout the paper. Second, if $h\neq0$, that global Z2 symmetry is explicitly broken. Thus, it would seem to me that neither the ferroelectric nor the antiferroelectric state correspond to genuine spontaneously broken symmetries when h\neq 0. How should one then understand the stability of the antiferroelectric phase? Is it because it also breaks translation symmetry while the ferroelectric state (and the SPT phase) does not? (By the way, the authors may consider using "ferroelectric" and "antiferroelectric" instead of FM and AFM when discussing those phases.)

Our reply

We would first like to clarify a mistake that might have confused the referee. On page 10 we incorrectly state that: "For the FM Ising interaction μτ < 0 we always obtain a disordered phase.". Instead the FM state in such regime remain stable upon introducing finite value of h, however the $\mathbb{Z}_2$ system immediately chooses the FM that is aligned with the sign of h. We corrected the sentence which now reads: "For the FM Ising interaction$\mu_\tau < 0$ the system remains a FM, and the field lifts the degeneracy between the two possible FM states, according to the sign of h [58].", where we added a new reference:

[58] A. Yuste, C. Cartwright, G. D. Chiara, and A. Sanpera, Entanglement scaling at first order quantum phase transitions, New Journal of Physics 20, 043006 (2018).

In the sense described by the referee the global symmetry in the AFM case is indeed explicitly broken for $h \neq 0$. The referee is also correct that in the AFM phase the system also breaks the translational symmetry. This is the key: even if the global $\mathbb{Z}_2$ symmetry is broken by $h \neq 0$, translational symmetry remains intact in the Hamiltonian, but is spontaneously broken in the AFM even for $h \neq 0$. We believe that by fixing the above sentences we clear the confusion.

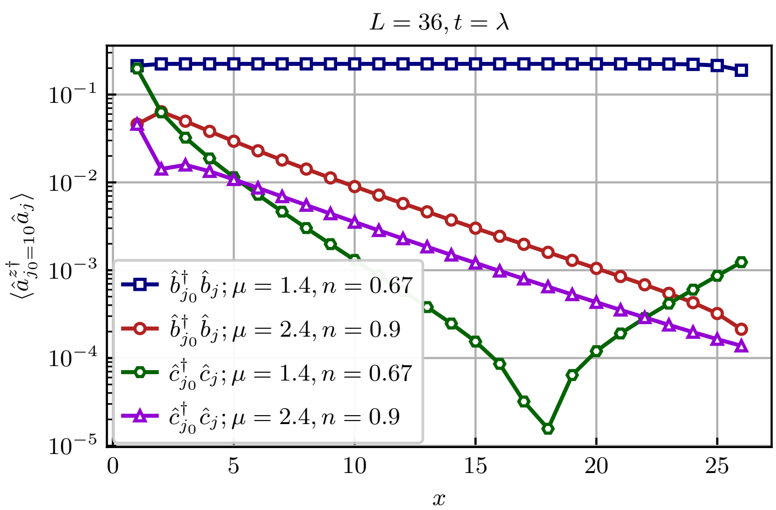
We added the sentence on page 3 "This term explicitly breaks the global Z2 magnetic symmetry for $h \neq 0$ [28].", to make it clear to the readers that the magnetic symmetry is explicitly broken for

$h \neq 0$. We believe a more detailed discussion is not needed and we refer the readers to the discussion in Borla et al. in Ref [28].

4. I am confused why the gauge-invariant Green's function (6) does not decay exponentially at h=0 when the pairing term \lambda is nonzero. As the authors show in Appendix A, in that limit the model simply maps onto the "ungauged" Kitaev chain (A4), and the 2-point function (6) maps onto the ordinary $$ **2-point function of the gauge-invariant $\mathbf{b},\mathbf{b}^\dagger$ bosons. The Kitaev chain describes a gapped superconductor, thus I would expect the Green's function to decay exponentially even in the topological phase.**

Our reply

Our numerical simulations involve hard-core bosons and not spineless fermions, which are considered in the conventional Kitaev chain, to which we ofter refer throughout the text. One can apply the Jordan-Wigner transformation which maps between the hard-core bosons and spinless fermions, and recover the same form of the Hamiltonian. This comes from the fact that all term in the Hamiltonian consist of operators acting on the nearest neighbor and the terms acquire a minus sign at most, as discussed in Appendix A 1. Hence both the bosonic and the better known fermionic Hamiltonian have the same eigenvalue spectrum and the observables that involve nearest-neighbor operators, like the on-site parton number operator, coincide. However, long-range correlators do not necessary agree as the Jordan-Wigner strings play an important role. To be more precise: $\langle \hat{b}_i^\dagger \hat{b}_j \rangle = \langle \hat{c}_i^\dagger \left(\prod_{i \leq l < j} e^{i\pi \hat{n}_l} \right) \hat{c}_j \rangle \neq \langle \hat{c}_i^\dagger \hat{c}_j \rangle$. The string operator between the fermionic operators, $\hat{c}_j$, changes the overall behavior for $|i-j| > 1$. We demonstrate this in the attached figure, where we directly simulate the hard-core bosons and spinless fermions on a short lattice $L = 36$ using TeNPy[1] https://tenpy.readthedocs.io/en/latest/ :
One can see that the bosons remain almost constant, and fermions decay exponentially in the topological regime. On the other hand both fermions and bosons decay exponentially in the trivial regime.



[1] J. Hauschild and F. Pollmann, Efficient numerical simulations with Tensor Networks: Tensor Network Python (TeNPy), SciPost Phys. Lect. Notes , 5 (2018), code available from https://github.com/tenpy/tenpy, arXiv:1805.00055.

In our case the two-point correlation function thus directly probes the confinement of charges, i.e., it probes how easy it is for the partons to move away from each other.

Finally, we note that we compute the correlator by mapping it to the spin model, see Eq. 16. When $\lambda = -t$, the spin model reduces to the transverse field Ising where spins align along the z -direction. Hence this value is thus almost constant and only contains some spin flips due to the projectors at the end of the strings $\propto (1 - \tau^x_{j_0-1,j_0}\tau^x_{j_0,j_0+1})$. We added additional note that the Green's function is computed via the spin operators: "We note that since we integrate out the matter by using the Gauss law and directly simulate the system by using the spin-1/2 model, we had to express the Green's function in terms of the spin operators; see Eq. (16) and the Appendices B and D.".

In Fig. 4(c,d) and Fig. 9(c,d), although this is just a suggestion, I wonder why the authors did not use a log-linear plot to more clearly demonstrate the exponential decay.

Our reply

We believe that a log-log scale better highlights the different behaviors between the power-law decay and the exponential decay, which is the key feature of the Green's function in the deconfined and confined regime, respectively. Similar log-log scales have also been used in previous works, for example by Borla et al., Phys. Rev. Lett. **124**, 2020. We believe that a log-linear scale would be useful when analyzing the rate of the exponential decay, as was done in Kebric et al., 2024, Phys. Rev. B **109**, 2024.

Fig. 11 and the Sec. IV.D that discusses it are somewhat confusing because they give the impression that the mean-field theory becomes exact in the h=0 limit, which is clearly not true since for example, the phase boundaries obtained from the mean-field entanglement entropy (Figs. 7-8) do not match those from the full theory (Figs. 2-3). The authors should consider whether Fig. 11 is really necessary, since it only represents "partial" phase diagrams (in gauge/matter sectors separately)

Our reply

Figure 11. summarizes the results of the mean-field theory that we develop for the $\mathbb{Z}_2$ LGT. We thus believe that it is an important figure which summarizes all of the important results. Due to the Gauss law, we can either integrate out matter, or the gauge degrees of freedom. In our numerics we integrated out the matter. We believe that these are not just partial phase diagrams, but are phase diagrams that we obtain in the two different pictures, that is after we decouple the gauge and matter degrees of freedom. For example in order to establish the mean-field phase diagrams in the gauge sector, we had to take into account the results obtained from the mean-field for the charge sector. In that way this is not a partial phase diagram. On the other hand the mean-field for the matter sector completely decoupled from the gauge fields and thus we did not use any results from the gauge sector to establish its phase diagram.

We clearly state that the mean-field theory in the charge sector does not correctly capture the phase diagram in the confined regime $h \neq 0$. We only claim that the mean-field theory in the charge sector corresponds to the exact $\mathbb{Z}_2$ LGT when $h = 0$, since it has the same Hamiltonian.

At the end of Sec. V.B, the authors write that "the mean-field theory correctly captures the qualitative features of the full Z2 LGT, especially when $\lambda \neq 0$, when it starts to match quantitatively." The latter statement contradicts Figs. 13(a) and (c), which show that the λ dependence of P is not captured by the mean-field theory.

Our reply

This was a writing mistake on our end, and we thank the referee for pointing this out. What we really meant was that there is a remarkable agreement for $h \neq 0$. However, the qualitative features: transition at $n > 0$ can indeed be seen for $\lambda \neq 0$, however there is no dependence on the filling as discussed earlier in the text. Since we do not directly compare the curves we dropped the last part in the sentence where we claimed quantitative agreement: "This again shows that the mean-field theory correctly captures the qualitative features of the full $\mathbb{Z}_2$ LGT, especially when $\lambda \neq 0$.~~, when it starts to match quantitatively.~~"

The most obvious "failure" of the mean-field theory is its inability to capture the filling dependence of the confinement phase boundaries. Can the authors speculate why mean-field theory does not capture this?

Our reply

The mean-field theory in the gauge sector simply contains a transverse field $\propto -g \sum_{\langle i,j \rangle} \hat{\tau}^z_{\langle i,j \rangle}$. As we demonstrate in the Appendix B the exact $\mathbb{Z}_2$ LGT maps to a transverse field Ising when $\lambda = -t$. Although the strength of the g depends on the filling n and λ, this does not change the fact that the Hamiltonian remains a simple transverse field with strength g, for any value of λ. Hence although the rate of the SC pair fluctuations increases in the exact model, which is manifested in the filling dependence as a function of λ, this is not captured in the mean-field model for the gauge field.

We added additional text explaining, why there is no dependence on the filling at the end of Sec. IV. D.: "This is due to the fact that in the mean-field theory the strength of the SC term λ, only changes the value of g in the transverse field term $\propto -g \sum_{\langle i,j \rangle} \hat{\tau}^z_{\langle i,j \rangle}$. Relating the mean-field theory to the exact mapping of the $\mathbb{Z}_2$ LGT to the spin-1/2 model, this means that the mean-field theory effectively remains in the regime where $\lambda = -t$; see Appendix B, for the mapping of the $\mathbb{Z}_2$ LGT to the spin-1/2 model."

At the beginning of Sec. V.A: "… for different values of the Z2 electric field in Fig. 12." I assume they mean different values of the pairing term λ? Relatedly, in the caption of Fig. 12, the authors could perhaps point out that the legend in (b) applies to both panels.

Our reply

We thank the referee for pointing out the unclear formulation in our text. We meant that Fig.12 was computed for different values of λ and h. We modified the sentence to make it more clear to: "… for different values of the pairing term λ and the $\mathbb{Z}_2$ electric field h in Fig. 12. ".

To make the Fig. 12 more clear we added the extra sentence in the caption: "Both legends apply for the two panels in the figure."

In several places (especially Appendix E, but also elsewhere), the authors use the expression "the Z2 electric field h". h itself is not the electric field, so they should write "the Z2 electric field term h" or "the Z2 electric string tension h".

Our reply

We agree with the referee. We added the word "term" correspondingly where it was necessary.

In Appendix A, the subscripts j,j+1 are not enclosed in <…> while they are elsewhere in the text.

Our reply

We fixed the typographical mistakes, and thank the referee for pointing them out.

Caption of Fig. 1: "anti-stings" --> "anti-strings"

Our reply

We fixed the typographical error.

List of changes:

- We added the following sentence to the beginning of section IV.: "We note that variational mean-field approach has been successfully used to study transitions in LGTs in higher dimensions [51–53].", where [51-53] are now: [51] S. D. Drell, H. R. Quinn, B. Svetitsky, and M. Weinstein, Quantum electrodynamics on a lattice: A Hamiltonian variational approach to the physics of the weak-coupling region, Physical Review D 19, 619 (1979). [52] D. Boyanovsky, R. Deza, and L. Masperi, Variational method for the Z(2)gauge model, Physical Review D 22, 3034 (1980). [53] D. Horn and M. Weinstein, Gauge-invariant variational methods for Hamiltonian lattice gauge theories, Physical Review D 25, 3331 (1982).

- We corrected the sentence in Sec. IV B. 1., which now reads: "For the FM Ising interaction $\mu_\tau < 0$ the system remains a FM, and the field lifts the degeneracy between the two possible FM states, according to the sign of h [58].", and we added a new reference: [58] A. Yuste, C. Cartwright, G. D. Chiara, and A. Sanpera, Entanglement scaling at first order quantum phase transitions, New Journal of Physics 20, 043006 (2018).

- We added the sentence on page 3 "This term explicitly breaks the global Z2 magnetic symmetry for $h \neq 0$ [28].",

- We added an additional remark on how we compute the $\mathbb{Z}_2$ invariant Green's function in our numerical calculations of the exact mode in Sec. III C. 1. : "We note that since we integrate out the matter by using the Gauss law and directly simulate the system by using the spin-1/2 model, we had to express the Green's function in terms of the spin operators; see Eq. (16) and the Appendices B and D.".

- We deleted the last part of the sentence in Sec. V B: "…, ~~when it starts to match quantitatively~~."

- Additional explanation was added at the end of Sec. IV. D.: "This is due to the fact that in the mean-field theory the strength of the SC term λ, only changes the value of g in the transverse field term $\propto -g \sum_{\langle i,j \rangle} \hat{\tau}^z_{\langle i,j \rangle}$. Relating the mean-field theory to the exact mapping of the $\mathbb{Z}_2$ LGT to the spin-1/2 model, this means that the mean-field theory effectively remains in the regime where $\lambda = -t$; see Appendix B, for the mapping of the $\mathbb{Z}_2$ LGT to the spin-1/2 model."

- We modified the sentence in Sec. V. A. to: "… for different values of the pairing term λ and the $\mathbb{Z}_2$ electric field h in Fig. 12. ".

- We added an additional sentence in the caption of Fig. 12: "Both legends apply for the two panels in the figure."

- We added an extra sentence on page 8 of the paper: "Similar pair-fluctuating mechanism is at work at high fillings, where the chain is almost completely filled, and the SC terms annihilate pairs, which are then again generated, before partons are able to hop."

- We added the word "bipartite" in the first sentence of the Section III. B 1., which now reads: "To determine the phase diagram of the model, we first consider the bipartite entanglement entropy

S(x) as a function of filling n, and the strength of the SC term λ at different Z_2 electric field strengths h."

- We add the sentence at the end of the first paragraph in Section III. B 1: "The bipartite entanglement entropy S(x) was extracted from the MPS directly from the singular value decomposition at a given bond x, which divides the system into subsystem A and B [33]."

- We corrected the typographical error on page 4, where we incorrectly labeled the "unitary transformation" as a "gauge transformation". The full sentence now reads: "Indeed, by a unitary transformation $\hat{a}_j \to e^{i\frac{\pi}{2}}\hat{a}_j$, the model Eq. (1) is equivalent at $\pm\lambda$, but this transformation has a non-trivial effect on the entanglement we calculate after integrating out matter fields, i.e., directly in the $\hat{\tau}^x_{\langle i,j\rangle}$ basis, see Appendix B. ".

- We fixed a typographical error in the last sentence of section III. D. Its correct form is now: "Finally, for intermediate densities the spontaneously symmetry-broken phase disappears and the phase diagram only contains an extended confined Higgs phase at $h, \lambda \neq 0$, in addition to the confined meson (parton) Luttinger liquid at any $h \neq 0$ (at the special point $h = 0$) when $\lambda = 0$."

- We fixed a typographical error on page 8:" $\mu < 2$" to "$\mu < -2t$".

- We fixed a typographical error in Eq. B3., which now reads:

$$\left(\prod_{l<j}\hat{\tau}^z_{l,l+1}\right)\hat{a}^\dagger_j = \left(\prod_{l<j}\hat{\tau}^z_{l,l+1}\right)\frac{1}{2}\left(1+\hat{\tau}^x_{j-1,j}\hat{\tau}^x_{j,j+1}\right) \text{ and } \left(\prod_{l<j}\hat{\tau}^z_{l,l+1}\right)\hat{a}_j = \left(\prod_{l<j}\hat{\tau}^z_{l,l+1}\right)\frac{1}{2}\left(1-\hat{\tau}^x_{j-1,j}\hat{\tau}^x_{j,j+1}\right).$$

- We added the sentence on the Green's function in Section III. C. 1. "We note that since we integrate out the matter by using the Gauss law and directly simulate the system by using the spin-1/2 model, we had to express the Green's func- tion in terms of the spin operators; see Eq. (16) and the Appendices B and D.".

- We fixed a typographical error on page 21 in section D 1., where the text in line 2.(b) now correctly reads: " If the target filling n is greater than the calculated filling $n(p = 1)$ we redefine $\mu_{min} = \mu_j$. In contrast if the target filling is smaller than the calculated filling we redefine $\mu_{max} = \mu_j$j.".

- We fixed other various typographical mistakes, most notably the missing "$\langle\rangle$" in equations and adding the word "term" in the expression, "$\mathbb{Z}_2$ electric field term h".

Source for the code used for the figure in this Referee reply:

[1]J. Hauschild and F. Pollmann, Efficient numerical simulations with Tensor Networks: Tensor Network Python (TeNPy), SciPost Phys. Lect. Notes , 5 (2018), code available from https://github.com/tenpy/tenpy, arXiv:1805.00055.