Dear Editor, Dear Referees,

we would like to thank the referees for their time and a thorough analysis of our manuscript. We are delighted about the positive comments made by the referees and are grateful for the many helpful comments. Below we directly address all of the comments made by Referee 2. We incorporated all of the suggestions and comments made by the referees, which we summarized at the end of this document. We believe that our manuscript is now ready for publication in SciPost.

Sincerely,
Matjaž Kebrič, on behalf of all authors

Report 2

Summary:

The paper studies the $(1+1)D$

quantum version of the Abelian Higgs model with discrete, Abelian gauge group, modeled by hardcore-bosons coupled to Z_2 gauge fields on the lattice, including a superconducting pairing term.
The authors present simulation results of the model, obtained via DMRG, and map out the phase diagram in the space of electric field strength h, filling n and coupling constant λ, thus re-deriving to a large extent the results on the phase diagram of Borla et al, 2021 (Ref.28 in the current version).
The work subsequently sheds more light on the confinement aspect of the theory using as observables the gauge invariant two-point function and string-length histograms.
As actual novelty the authors solve the model using a mean-field, self-consistent approach, starting with a product-state ansatz for the wavefunction. The phase diagram obtained in mean-field theory is subsequently compared to the full theory finding good agreement in many but not all aspects.

General remark:

The paper is clearly written and gives a good account of the theory underlying the model. Ultimately, the work lives up to its title and delivers the mean-field solution to the theory under study. In addition, the phase diagram of the full theory is revisited with different observables, thus shedding some more light on the microscopic mechanisms at work. The authors discuss at length these aspects of the phase diagram found in the full theory via DMRG and using a mean-field approach. Even though the mean-field, variational approach for LGTs in the Hamiltonian formalism is a well-known method, the paper represents original research and has a sufficient degree of novelty.

Our reply

We would like to thank the referee for their report and summary of our work. We are pleased that the referee finds this work interesting and novel.

Requested changes

1) The dynamical confinement mechanism for the theories with $(h=0,\lambda\neq0)$

and $(h>0)$ seems to be different. In particular, the authors offer an explanation for the theory at low filling and $(h=0,\lambda\neq0)$ where partons are basically created in pairs, separated by one unit of the lattice spacing which thus results in the corresponding string-length histogram which is peaked at $l=1$. What about at high filling for the same theory $(h=0,\lambda\neq0)$?

Our reply

The confining mechanism for high filling is the same as for low filling. Also there particle fluctuations come in pairs, since they are generated by the SC term $\propto \lambda$. At high fillings we have pair fluctuations where mesons are annihilated and then again created. This process can again only occur before a meson vacancy could be filled by other partons. Hence pair fluctuations are effectively confined in the same manner as at low filling.

Such confining feature is shown in the Green's function which decays exponentially also for high fillings in Fig. 4(b). String-length histograms can, however, not be used as an effective probe of confinement at high fillings, as the average distance between mesons becomes smaller. In the limit of unity filling, $n \to 1$, it approaches $\ell \to 1$. Hence at high filling $n \gtrsim 0.9$, where we see exponential decay of the Green's function, we can not see the bimodal distribution which is key signature of confinement at lower fillings.

To make this clearer we added an additional sentence on page 8 of the paper, which reads:
"Similar pair-fluctuating mechanism is at work at high fillings, where the chain is almost completely filled, and the SC terms annihilate pairs, which are then again generated, before partons are able to hop."

2) Sect. 3B.1: The entanglement entropy $S(x)$ is used here but never formally defined in the work.

Our reply

We thank the referee for pointing this out. In the manuscript we state that we compute the bipartite entanglement entropy $S(x)$, with a cut on site x, which divides the system into two subsystems A and B, see Section III. B 1., page 4. In our numerical simulations, we integrate out the matter degrees of freedom, by using the Gauss law, and thus directly simulate only the gauge degrees of freedom, which are represented by the Pauli matrices. We calculate the bipartite entanglement entropy which is encoded in the matrix-product state (MPS) bond between sites $\tau_{x-1,x}$ and $\tau_{x,x+1}$. More precisely, the von Neumann entanglement entropy can be calculated directly from the singular value decomposition (SVD) as $S(A\,|\,B) = -\sum_a s_a^2 \log_2(s_a^2)$, where s_a are the singular values, see for example Ref. [33], Schollwöck, The density-matrix renormalization group in the age of matrix product states, Annals of Physics 326, 96 (2011).

To make this a bit clearer we added the word "bipartite" in the first sentence of the Section III. B 1., which now reads:
"To determine the phase diagram of the model, we first consider the bipartite entanglement entropy S(x) as a function of filling n, and the strength of the SC term λ at different Z_2 electric field strengths h."
In addition, we added the following sentence at the end of the first paragraph in Section III. B 1.:
"The bipartite entanglement entropy S(x) was extracted from the MPS directly from the singular value decomposition at a given bond x, which divides the system into subsystem A and B [33]."

3) Sect. 3B.1, in the discussion relating the theories at $\lambda=1$ and $\lambda=-1$: "Indeed, by a gauge transformation ..." This should read "unitary transformation", as discussed in the appendix, since the model is only gauged under $Z_2=\{1,e^{\pm i\pi}\}$.

Our reply

We thank the referee for carefully reading the manuscript and catching this writing error. Indeed, the correct formulation would be to write “unitary transformation”. We corrected the text which now reads:

“Indeed, by a unitary transformation $\hat{a}_j \to e^{i\frac{\pi}{2}}\hat{a}_j$, the model Eq. (1) is equivalent at $\pm\lambda$, but this transformation has … ”

4) Sect. D, towards the end of the section: "... in addition to the deconfined meson (parton)..." What is meant is the confining phase or meson LL.

Our reply

We thank the referee for pointing out this typo. The meson Luttinger liquid is indeed the confined phase as partons are confined into meson, and the parton Luttinger liquid is simply the regime where partons are completely free and form a non-interacting Luttinger liquid. We corrected the typographical error and the revised text now reads: “… to the confined meson (parton) Luttinger liquid at any …”.

5) Fig.4 and Fig.9: Displaying the correlators with log-linear scale would perhaps be beneficial.

Our reply

The same comment was given also by the referee in the first report. We repeat the answer again here:
We believe that a log-log scale better highlights the different behaviors between the power-law decay and the exponential decay, which is the key feature of the Green’s function in the deconfined and confined regime, respectively. Similar log-log scales have also been used in previous works, for example by Borla et al., Phys. Rev. Lett. **124**, 2020. We believe that a log-linear scale would be useful when analyzing the rate of the exponential decay, as was done in Kebric et al., 2024, Phys. Rev. B **109**, 2024.

List of changes:

- We added the following sentence to the beginning of section IV.: "We note that variational mean-field approach has been successfully used to study transitions in LGTs in higher dimensions [51–53].", where [51-53] are now: [51] S. D. Drell, H. R. Quinn, B. Svetitsky, and M. Weinstein, Quantum electrodynamics on a lattice: A Hamiltonian variational approach to the physics of the weak-coupling region, Physical Review D 19, 619 (1979). [52] D. Boyanovsky, R. Deza, and L. Masperi, Variational method for the Z(2)gauge model, Physical Review D 22, 3034 (1980). [53] D. Horn and M. Weinstein, Gauge-invariant variational methods for Hamiltonian lattice gauge theories, Physical Review D 25, 3331 (1982).

- We corrected the sentence in Sec. IV B. 1., which now reads: "For the FM Ising interaction $\mu_\tau < 0$ the system remains a FM, and the field lifts the degeneracy between the two possible FM states, according to the sign of h [58].", and we added a new reference: [58] A. Yuste, C. Cartwright, G. D. Chiara, and A. Sanpera, Entanglement scaling at first order quantum phase transitions, New Journal of Physics 20, 043006 (2018).

- We added the sentence on page 3 "This term explicitly breaks the global Z2 magnetic symmetry for $h \neq 0$ [28].",

- We added an additional remark on how we compute the $\mathbb{Z}_2$ invariant Green's function in our numerical calculations of the exact mode in Sec. III C. 1. : "We note that since we integrate out the matter by using the Gauss law and directly simulate the system by using the spin-1/2 model, we had to express the Green's function in terms of the spin operators; see Eq. (16) and the Appendices B and D.".

- We deleted the last part of the sentence in Sec. V B: "…, ~~when it starts to match quantitatively~~."

- Additional explanation was added at the end of Sec. IV. D.: "This is due to the fact that in the mean-field theory the strength of the SC term λ, only changes the value of g in the transverse field term $\propto -g \sum_{\langle i,j \rangle} \hat{\tau}^z_{\langle i,j \rangle}$. Relating the mean-field theory to the exact mapping of the $\mathbb{Z}_2$ LGT to the spin-1/2 model, this means that the mean-field theory effectively remains in the regime where $\lambda = -t$; see Appendix B, for the mapping of the $\mathbb{Z}_2$ LGT to the spin-1/2 model."

- We modified the sentence in Sec. V. A. to: "… for different values of the pairing term λ and the $\mathbb{Z}_2$ electric field h in Fig. 12. ".

- We added an additional sentence in the caption of Fig. 12: "Both legends apply for the two panels in the figure."

- We added an extra sentence on page 8 of the paper: "Similar pair-fluctuating mechanism is at work at high fillings, where the chain is almost completely filled, and the SC terms annihilate pairs, which are then again generated, before partons are able to hop."

- We added the word "bipartite" in the first sentence of the Section III. B 1., which now reads: "To determine the phase diagram of the model, we first consider the bipartite entanglement entropy

S(x) as a function of filling n, and the strength of the SC term λ at different Z_2 electric field strengths h."

- We add the sentence at the end of the first paragraph in Section III. B 1: "The bipartite entanglement entropy S(x) was extracted from the MPS directly from the singular value decomposition at a given bond x, which divides the system into subsystem A and B [33]."

- We corrected the typographical error on page 4, where we incorrectly labeled the "unitary transformation" as a "gauge transformation". The full sentence now reads: "Indeed, by a unitary transformation $\hat{a}_j \to e^{i\frac{\pi}{2}}\hat{a}_j$, the model Eq. (1) is equivalent at $\pm\lambda$, but this transformation has a non-trivial effect on the entanglement we calculate after integrating out matter fields, i.e., directly in the $\hat{\tau}^x_{\langle i,j\rangle}$ basis, see Appendix B. ".

- We fixed a typographical error in the last sentence of section III. D. Its correct form is now: "Finally, for intermediate densities the spontaneously symmetry-broken phase disappears and the phase diagram only contains an extended confined Higgs phase at $h, \lambda \neq 0$, in addition to the confined meson (parton) Luttinger liquid at any $h \neq 0$ (at the special point $h = 0$) when $\lambda = 0$."

- We fixed a typographical error on page 8:" $\mu < 2$" to "$\mu < -2t$".

- We fixed a typographical error in Eq. B3., which now reads:

$\left(\prod_{l<j}\hat{\tau}^z_{l,l+1}\right)\hat{a}^\dagger_j = \left(\prod_{l<j}\hat{\tau}^z_{l,l+1}\right)\frac{1}{2}\left(1+\hat{\tau}^x_{j-1,j}\hat{\tau}^x_{j,j+1}\right)$ and $\left(\prod_{l<j}\hat{\tau}^z_{l,l+1}\right)\hat{a}_j = \left(\prod_{l<j}\hat{\tau}^z_{l,l+1}\right)\frac{1}{2}\left(1-\hat{\tau}^x_{j-1,j}\hat{\tau}^x_{j,j+1}\right).$

- We added the sentence on the Green's function in Section III. C. 1. "We note that since we integrate out the matter by using the Gauss law and directly simulate the system by using the spin-1/2 model, we had to express the Green's func- tion in terms of the spin operators; see Eq. (16) and the Appendices B and D.".

- We fixed a typographical error on page 21 in section D 1., where the text in line 2.(b) now correctly reads: " If the target filling n is greater than the calculated filling $n(p = 1)$ we redefine $\mu_{min} = \mu_j$. In contrast if the target filling is smaller than the calculated filling we redefine $\mu_{max} = \mu_j$j.".

- We fixed other various typographical mistakes, most notably the missing "$\langle\rangle$" in equations and adding the word "term" in the expression, "$\mathbb{Z}_2$ electric field term h".