Reply for proofs of **First Referee** for

Diffusive hydrodynamics of hard rods from microscopics

Dear referee, we would like to thank you for your positive review and for the recommendation to publish. We also would like to thank you for the useful comments that improved the quality of the paper, and in particular its readability. In the following we comment the changes you requested.

- *The new correction to the Euler GHD of hard rods is being called diffusive, although from equation (9) it is not obvious. However, its form in local equilibrium (eq. (3)) makes this clear. It would be great to clarify this point.*
  This is a good point. The new correction is an equation governing the diffusive scale, but it is actually not a diffusive equation. To avoid confusion we now call it "diffusive scale equation" and also added a comment in the introduction and the conclusion.

- *The set of equations (9) and (55) being time-reversal symmetric, they do not show an 'arrow of time'. Has this also been tested numerically with the definition of entropy in eq. (38) which does not include two-point correlation in its definition.*
  A simple way to obtain a scenario where the entropy as in eq. (39) decreases is to evolve an initial local equilibrium state, with equations (9) and (55), forward in time (this will typically increase entropy because, for local equilibrium states, eq. (10) reduces to the entropy increasing eq. (3)) and then apply time-reversal symmetry, i.e. $p \to -p$. This new state will then evolve exactly backwards in time, hence decreasing entropy.

- *The correction to the average position $\langle X_e(t)|x_1, p_1\rangle$ of the quasiparticle tagged by the initial position and velocity (x_1, p_1) receives two corrections at $O(1/\ell)$: one from initial fluctuations and the other from long-range correlation. The former contribution is easy to understand. Is it possible to understand the second mechanism physically?*
  The two contributions are due to the instantaneous **singular** part of the fluctuations (which are not necessarily the initial fluctuations) and the instantaneous long range correlations. The singular part represents fluctuations inside a fluid cell, the long range part correlations across fluid cells. Indeed, the first contribution gives rise to the NS term in Eq. (8). The physical picture behind the second mechanism is that the velocity of particles depends on the presence of other particles, see eq. (2). In tendency, the larger the density of particles, the larger the effective velocity (since particles 'jump' more often, see for example Figure (1)). Thus, if say neighboring fluid cells are positively correlated, then if the particle travels faster through the first fluid cell it will likely also travel faster through the second one. This leads to an overall position shift.

  Also, does the first contribution give rise to the Kubo diffusion (NS term) in Eq. (7)?
  Yes, indeed, the first contribution give rise to the Kubo diffusion (NS term) in the equation. This is highlighted below Eq.48: ", the second term in RHS comes from the local GGE correlations and coincides, as expected, with the usual diffusion matrix in a GGE state". We now also present this important observation in the introductory section 1.1, below Eq.(8).

  Is the exact cancellation of the NS term and the $C^n_{\mathrm{LR,asym}}$ term universal? If possible, it *would be useful to make some intuitive comment on why such an exact cancellation occurs.* We believe that the cancellation is indeed universal for any integrable model. This is our proposal explained in [Phys. Rev. Lett. 134, 187101], where it arises due to the specific way observables need to be regularized in the coarse-grained hydrodynamic theory. This work can be seen as an independent (and more solid) check of this proposal. In order to put more light on this important aspect, we now stress it on Remark 4.

- *Page (9) after Eq. (13): In order to obtain a macroscopic forward derivative in Eq. (9), one has to evaluate the RHS of eq. (9) at $t = 0^+$ in macroscopic scale. This calculation has also been shown in eq. (D.20). Is it feasible to get the same solution by solving the coupled Eqs. (7) and (55) starting from a local GGE state, instead of the alternative computation presented in Appendix D?*
  Yes, this calculation is feasible. In fact in the companion paper [Phys. Rev. Lett. 134, 187101] from the same authors, the analogous calculation is made (for a generic integrable model) by solving the hydrodynamics coupled equations starting from a local GGE state. Otherwise, in this work, we presented only the calculation proposed in Appendix D being only based on hard rods microscopics.

- *Would one find such a 'diffusion' term originating from C^n_{LR} in non-integrable systems as well that support ballistic transport?*
  Yes, as we presented in the companion paper [Phys. Rev. Lett. 134, 187101], and as is further elaborated in the recent paper [arxiv:2506.05279], a similar calculation is valid more generically for systems with linearly degenerate hydrodynamics. In [arxiv:2506.05279] it is in fact shown that both a NS term, controlled by a projected Kubo formula, and a long-range term appear. This class of models includes integrable ones, as well as non-integrable examples. We already stressed this point in conclusions, at page 19, and we added a reference this recent work: "Similar equations like (52) can also be derived more generally based on general hydrodynamic principles for models with linearly degenerate hydrodynamics (which includes integrable systems), see our companion paper [1] and the recent work [30].". For non-integrable models whose hydrodynamics is not linearly degenerate it is well known that their leading order correction is superdiffusive (we added a footnote in the conclusion), which dominates over the diffusive scale.

- We have corrected the highlighted typos.

- *I feel moving the discussion on X(t, x1, p1) along with Eqs. (34) and (35) immediately after Eq. (30) would be better for the readers.*
  We implemented this comment.

- *p12, after Eq. (38): I feel it would be useful for the reader to elaborate the following statement a little more: "However, because the structure of the state remains invariant under time and the hard rods dynamics do not have any memory, this immediately implies that (38) holds at all times."* We have adapted the statement, explaining better why we believe that the equation holds at all times.

Kind regards
Friedrich Hübner, Leonardi Biagetti, Jacopo De Nardis and Benjamin Doyon

Reply for proofs of **Second Referee** for

Diffusive hydrodynamics of hard rods from microscopics

Dear referee, thank you very much for your positive comments, that we think can visibly improve the quality of this work. We took your comments seriously and tried to compensate all the weak points highlighted in your report.

As a general clarification, we are very aware that we are studying a highly artificial model (at least in the context of hydrodynamics). Integrable models, such as hard rods, have an infinite number of (local) conserved quantities and thus notions like thermal states ($\rightarrow$ GGE) and hydrodynamics ($\rightarrow$ GHD) have to be adapted to include these additional charges. Therefore, it is clear that the hydrodynamic equation cannot be exactly the Navier-Stokes equation (as this assumes only 3 conserved quantities – energy, momentum and particle number). However, there exists phenomenological derivations of the Navier-Stokes equation based on local equilibrium, which for integrable models like hard rods is given by a GGE. Therefore, it is natural to expect that the version of the Navier-Stokes eq. (3) correctly describes the diffusive correction. In addition, recall that eq. (3) had been rigorously proved in hard rods at infinitesimal macroscopic time from a class of local-equilibrium states, see [Comm. in Math. Physics 189(2), 577 (1997)]. The problem with the statement in [Comm. in Math. Physics 189(2), 577 (1997)] is that it indeed only holds at infinitesimal macroscopic times, $t \rightarrow 0^+$. This was believed to be sufficient as thermalization would drive the system back to local equilibrium. The main message of this paper is that this "driving back to local equilibrium" does not happen, as memory about long range correlations is preserved. This is an aspect that, we believe, has not been emphasised sufficiently in the past in explanations about Navier-Stokes and other hydrodynamic corrections, and hence this observation might be potentially more generally relevant.

- *In this integrable model system, the initial velocity distribution is preserved at least without external forces. In view of the according lack of thermalization (if starting from a non-Maxwellian velocity distribution) and ergodicity, it is perhaps not surprising that to first order in 1/l the diffusive Navier-Stokes behavior is not recovered. In the paper, the authors should elaborate in some detail on this point. It seems to me that thermalization and time irreversibility are absent in general even when (all) higher order powers in* $1/\ell$ *are considered.*
  See above: we consider a version of Navier-Stokes that accounts for the additional conserved quantities of hard rods. This version had been proven to be correct for $t \rightarrow 0^+$ in local equilibrium states, and assumed to be correct at all times, implying irreversibility based on generalised thermalisation.

- *In the context of their present work, can the authors comment on the single-filing constraint for the one-dimensional rod system, which causes a sublinear (fractional) time dependence of the mean-squared displacement of the rod centers at long times?*
  We expect that you are referring to the trajectories of physical hard rods, which exchange momenta during collision, but keep their order. Here, instead we consider what is called tracer dynamics. At collision particles keep their momenta, but exchange their positions. Note that this just amounts to a suitable relabeling of particles at each collision, hence it does not affect any hydrodynamic quantity (which are invariant under particle label permutations). As opposed to physical hard rods, the trajectories of tracer hard rods are

ballistic. The relation of our results with the physical rods' trajectories is an interesting question, but is beyond the scope of this work.

- *What precisely is the relevance of the present work regarding the (hydro-)dynamics of more realistic Newtonian hard-disk and hard-sphere systems in two and three dimensions, respectively, where single-filing is absent?*
  Since hard spheres in one-dimension are much different from hard spheres in higher dimension, there is not direct lesson of this work for higher dimension. However, we think that it shows that the subtle assumptions done to derive hydrodynamic equations should be carefully revisited, also in higher dimension. Note that the effect of long range correlations will also be present in higher dimension (of order one over volume scale), but will be subleading to a diffusive term (one over length scale). Hence, only in one dimension we expect that the long range correlations have an effect on the leading order correction.

- *In the line following Eq. (4), ℓ is defined quite vaguely as the ratio between macroscopic and microscopic scales. A more precise definition is desirable.*
  The definition of ℓ is was intuitive and informal – yet it is in fact a precisely defined parameter, making our statements accurate. We have added a sentence after eq 4 in the introduction to make it more precise. A more mathematical setup, which is the basis of our derivation, is as follows: consider a family of hard rods initial states labeled by ℓ. This family is such that $\langle \rho_e(x,p) \rangle$, as in eqs (17,18), approaches a smooth function as $\ell \to \infty$ and two-point correlation functions times ℓ approach smooth functions, while higher order correlation functions decay faster than $1/\ell$, eq (19). Thus mathematically speaking, ℓ is an external parameter setting the scale of the initial state and of the observation space-time positions (note that microscopic length scales decay as $1/\ell$, as in eq (16)). In particular, the final equations are expansions in $1/\ell$. The connection to the physical perspective is as follows: consider an experiment with a large number of particles. Then ℓ cannot – and does not need to – be defined unambiguously for our results to hold; different definitions give strengths of the correction terms. A good choice of ℓ, which minimises correction terms, is such that, in the initial state, on a distance scale ℓ there are a large number of particles, while variation lengthscales of local observables in this state are of an order-unity proportion of ℓ.

- *Second line of Section 1.1: What is the physical meaning of the annotated 'quasiparticles'?*
  We thank the author for this comment, we now introduce the definition of quasiparticles already at the beginning of section 1.1. We also now rephrase Remark 2 in order to connect it to the definition of "quasiparticles". The quasiparticles are the particles that undergo tracer dynamics (see 3 comments above). They are not the physical particles. Note that the notion of quasi-particles exists more general in integrable model and is a fundamental concept in GHD (hence the unifying name quasi-particle density).

- *Eq. (5) includes the 'empirical' density $\rho_e(t,y,q)$ whose definition, however, is postponed to Eq. (17). What precisely is the intended meaning of 'empirical'?*
  Empirical density is the observed density of a single realization of the stochastic sample (as opposed to the averaged density). It is a common concept in probability theory. In order to make the discussion more clear, we now introduce the definition of the empirical density in Eq.7, also stressing its physical interpretation.

- *Can the authors assign a physical meaning to the jump at $x = y$ of the long-range correlation function in Eq. (13) and Fig. 2 ?* Unfortunately, we do not have a good intuition about the physical meaning of the jump. In [Phys. Rev. Lett. 134, 187101] we show that it has to be mathematically there to compensate the change of the singular $\delta(x - y)$ part of the correlations (which follows the GGE correlations) in time, i.e. the only way to compensate the $\delta(x - y)$ is to have a space derivative act on a $\mathrm{sgn}(x - y)$. So they are in some sense generated by the change of local GGE correlations. Also, the jump of the correlations appears only for modes that have different velocities. Based on that, a natural interpretation could be that the change of local GGE correlations generates long range correlations, which depending on the velocity either affect the fluid cell to the left or the right (depending on their velocity). However, we do not find it very clear how to connect this picture with the explicit expression (equation 14 in the new version).

- We have corrected the highlighted typos.

Kind regards

Friedrich Hübner, Leonardi Biagetti, Jacopo De Nardis and Benjamin Doyon

Reply for proofs of **Third Referee** for

Diffusive hydrodynamics of hard rods from microscopics

Dear referee, thank you very much for your comments, that we think can substantially improve the quality and readability of our work. We regret for the inaccuracies, which we address in the revised version of the manuscript. In the following we comment the changes you requested.

- *Page 4, Section 1.1: In stating the main results, the definition of the dynamics and the notation should be given beforehand. For example, in (5) what is ρ_e? How does this correlation depend on ℓ?*
  We implemented this comment by introducing the definition of ρ_e already in Section 1.1 (Eq.7).

- *Page 7, Eq. (14): This is not a probability measure on the configuration space and cannot be normalized except under overly strong conditions on β. You probably mean a Poisson field with intensity $e^{-\beta}$, which should be stated precisely. (In Section 4 you mention this point correctly.)*
  We assume that $\beta(x,p)$ is chosen such that it is a normalizable probability measure. In particular, this means that $\beta(x,p) \to \infty$ as $x \to \pm\infty$ sufficiently fast (for instance linear growth would be fine).

- *Page 7, 3rd line after (17): Who is ℓ^0? Do you mean terms of order O(1)?*
  Yes, indeed. In order to improve the clarity we changed with $a = O(1)$ in this line.

- *Page 7, Eq. (18): I do not see why you call this "large deviation scaling"; it appears instead to be a condition on the correlations of the fluctuation fields.*
  An extensive quantity $A = A(\ell)$ satisfies the large deviation principle, if $F[\lambda] = \lim_{\ell\to\infty} \frac{1}{\ell} \log\langle e^{\lambda A}\rangle$ is finite (plus mathematical technicalities). From this it follows (if $F[\lambda]$ is smooth) that all cumulants scale as $\langle A^n\rangle^{\mathrm{c}} = \mathcal{O}(\ell)$. Its density $a = A/\ell$ then satisfies $\langle a^n\rangle^{\mathrm{c}} = \mathcal{O}(\ell^{1-n})$. This is what we call the "large-deviation scaling" for a density. Similar results exist for multivariate cumulants. Considering $\{\rho_{\mathrm{e}}(x,p)\}_{x,p\in\mathbb{R}}$ as an infinite collection of random variables, each of density type, then eq (19) is their formal large deviation scaling. We emphasise that Eq 19 is a weak condition; typical state satisfy it, see e.g. [26].

- *Page 8, Assumption 2: If you include in the statement "In particular, this is satisfied for all time t...", it seems to be part of the assumption, whereas in Appendix D you show that it follows from the dynamics.*
  Indeed, if the evolution is started from a local equilibrium state (the canonical choice in hydrodynamics) then assumption 2 is always satisfied. However, our result also applies to situations where the evolution is started from other states, as long as assumption 2 holds.

- *Page 8, Assumption 3: I do not understand this assumption. Correlations are always symmetric by definition. What is meant by "the microscopic shape"?*
  Here we mean symmetry with respect to x. On the microscopic scale the two point correlation function is not given by a simple delta peak. Instead, it has the well known shape presented in Eq. (F.1) and (F.2). However, on the macroscopic scale, this shape is replaced

by a Dirac delta function, whose weight corresponds to the total integral of the microscopic shape.

- *Page 12: It is not clear why (36) and (32) imply (37), since derivatives in x are involved in (32). As the work is not a rigorous proof, this could perhaps be introduced as a reasonable assumption.* We will use new equation numbers in this response: Mathematically speaking, equation (35) should be interpreted as (34), i.e. in a distributional sense, where the derivatives act on the test function. Using (37) this implies (38) quite generally (bounded and sufficiently fast decaying, so that one can exchange limit and integrals).

- *Page 16, Eq. (60): With this density, the positions are not distributed according to a homogeneous Poisson process, as stated. The distances between them are therefore not identically distributed, but depend on their locations (they are rather localized under the density (60)).*
  In general, you are correct: the positions of hard rods for a local equilibrium state are not distributed like a Poisson process. However, in the special case that $\int \mathrm{d}p\, \rho(x,p)$ is constant in x (as in our case), this is a known (but non-trivial) result [Comm. in Math. Physics 189(2), 577 (1997)].

Kind regards,
Friedrich Hübner, Leonardi Biagetti, Jacopo De Nardis and Benjamin Doyon