Dear Reviewer,
We sincerely thank you for your careful reading of our manuscript and for your positive evaluation.

We have carefully addressed all your comments and suggestions. In the following, we provide detailed responses to each point. For clarity, we highlight the reviewer's comments in blue and our responses in black.

We believe that the revised manuscript has been significantly improved and is now suitable for publication in SciPost Physics Core.

Sincerely,
Michele Viscardi, Marcello Dalmonte, Alioscia Hamma, Emanuele Tirrito

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List of Changes

- line 261: added definition of d and d_A;
- lines 273-286: improved the clarity of the paragraph;
- added footnote 3 and lines 528-529 to specify our bipartition of choice for the spectral quantities mentioned in the manuscript;
- lines 622-632: added a clarification about the magic localisation phenomenon in the quantum XY model;
- updated the references;
- line 383: specified the meaning of d;
- lines 517-526: clarified parameters of the XXZ Hamiltonians and the characteristics of its phase diagram;
- added footnote 4;
- removed Figure 8b.

Answer to Reviewer

> In this paper, the authors present a comprehensive study of nonstabilizerness in various spin-1/2 quantum spin chains. The authors numerically compute the stabilizer Rényi entropy (SRE) and the antiflatness of entanglement spectra to demonstrate their effectiveness in distinguishing different phases of matter. Specifically, the authors provide detailed analyses of the XXZ model, the XY model with and without Dzyaloshinskii-Moriya interactions, the cluster Ising model, and the cluster XY model. Overall, I believe the manuscript constitutes an important contribution to the field and merits publication in SciPost Physics Core.
>
> The paper is well-organized and includes a useful pedagogical review of the research area in the introduction and at the beginning of sections. To strengthen the content and improve readability, I request several improvements in the presentation of the manuscript as outlined below.

We thank the Referee for the positive assessment of our work and the support for its publication.

> (lines 372 ∼) The character d is defined differently from the previous subsection, where it was originally defined as the dimension of the local Hilbert space. Here, it should represent the dimension of the total Hilbert space, i.e., $d = 2^n$ for n-qubit systems.

We agree with the reviewer's comment and accordingly updated the manuscript immediately after Eq. 29 (line 383), where we explicate the different meaning of d.

> (lines 511 ∼) The explanation of the phase diagram for the XXZ model is somewhat confusing. In the definition of the Hamiltonian (Eq. (32)), the constant J is introduced. However, the authors give the phase transition lines as $h_s = 1+\Delta$and h_c, which should actually depend on J. I request revision to clarify the discussion in this section. Additionally, if the spin chain is subject to periodic boundary conditions, this should be explicitly stated.

We thank the reviewer for the comment. We updated the text according with their suggestions (lines 517-526), reconciling the Hamiltonian in Eq. 32 with that of the XXZ model in Ref. [1]. The expression of h_s is now written in full generality, including the parameter J, and the boundary conditions are the same as those in Eq. 32 (periodic).

> (lines 519 ∼) I do not understand the sentence "In the ferromagnetic phase $\Delta < 1$, the SRE is minimal,..." because in Fig. 3(a) the SRE appears to be minimal in the region $0 < \Delta$. Perhaps I have misunderstood something, or the horizontal axis in the plots is labeled incorrectly.

We thank the reviewer for spotting this typo. The updated Hamiltonian in Eq.32 is now in agreement with the plots in Fig. 3, where for $h = \frac{1}{2}, \Delta > 0$ we are in the ferromagnetic phase of the model.

> (lines 549 ∼) The authors assume open boundaries for the XY model, but is the classical-like ground state still obtained for open chains? Ref. [134] did not specify the boundary conditions in the model, but later assumed translational symmetry, which appears to be valid only for periodic chains.

We thank the reviewer for raising this point. The ground state (GS) of the quantum XY model in open boundary conditions (OBC) on the separability circle will, in general, not be completely factorised, due to the absence of the boundary term. Considering the latter as a local perturbation in the Periodic Boundary Conditions (PBC) Hamiltonian, the GS in OBC is expected to match that of the PBC Hamiltonian in the thermodynamic limit.

At finite sizes, we can quantify how factorised the ground state (GS) is by evaluating its average single-spin tangle $\langle\tau\rangle$. Let us notice that tangle is a rather standard quantity in entanglement theory and, in particular, it has been adopted for studying factorisation conditions for ground states in Ref. [2]. A pure state is fully factorised if and only if $\langle\tau\rangle = 0$, which means that each single-spin reduced density matrix ρ_i corresponds to a pure state (for spin-$\frac{1}{2}$ systems). The single-spin tangle τ_i of the i-th spin in the ground state $|GS\rangle$ is defined as [2]:

$$\tau_i = 4\,\mathrm{Det}\,\rho_i\,, \qquad \rho_i = \mathrm{Tr}_{\neq i}(|GS\rangle\langle GS|) \tag{1}$$

with N being the length of the spin chain and ρ_i being the i-th spin reduced density matrix of the state $|GS\rangle$. Similarly, the average single-spin tangle $\langle\tau\rangle$ of the ground state $|GS\rangle$ of a N-qubit system is defined as:

$$\langle\tau\rangle = \frac{1}{N}\sum_{i=1}^{N}\tau_i \tag{2}$$

We explore the average single-spin tangle $\langle\tau\rangle$ numerically, and we see it decreasing in the system size as $O(1/N)$ (see Fig. 2). The variance of the single-spin tangles $\mathrm{Var}(\tau)$ is also numerically shown to be

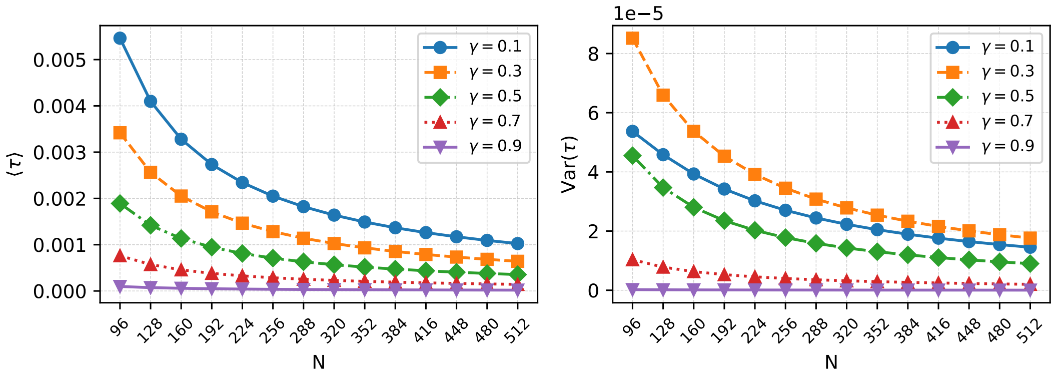

Figure 1: Average single-spin tangle $\langle\tau\rangle$ (left plot) and variance of the single-spin tangles $\mathrm{Var}(\tau)$ (right plot) for the GS of the quantum XY model on the separability circle with respect to the system size N and anisotropy parameters γ.

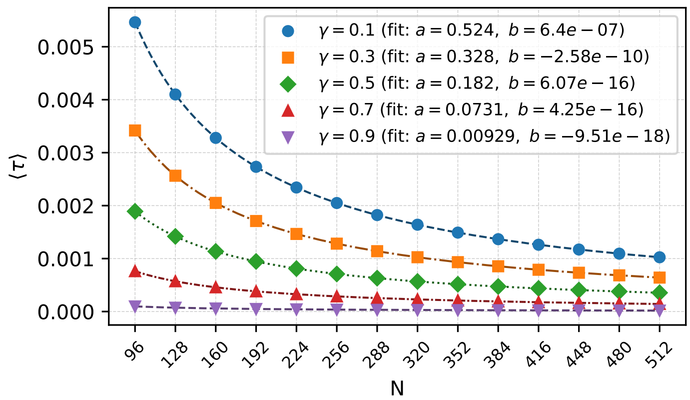

Figure 2: Average single-spin tangle, $\langle\tau\rangle$, (markers) and corresponding fits (dashed and dotted lines) of the form $a/N+b$ for the ground state of the quantum XY model along the separability circle, shown with respect to the system size N and anisotropy parameter γ.

decreasing. This behavior suggests not only that the average single-spin tangle becomes vanishing as N increases, but that being vanishing is typical in the chain (a sign of the GS approaching a fully factorised state as the system size increases). We report the results on the average tangle and the variance of tangles in the first figure.
As a final remark, we note that, as shown in Fig. 5b, the curves of magic on the separability circle have already converged for systems of size 64–128 spins, matching the theoretical values reported in Fig. 4.
Based on the Reviewer's question, we added a footnote at page 17 where we clarify both the existence of classical-like ground states in OBC and the convergence of the 2-SRE.

> (page 20 Fig. 8) The legends for the plots are illegible. It would be better to insert the equations manually in the plot area. Another minor comment is that the fitting curves should be moved to the background so that the data points appear in the foreground.

We agree with the comment, and updated the plots by moving the curves to the background and reporting the formulas for the fitted curves in a clearer format. To better extract the scaling of the quantities of interest, we increased the system size up to 704 spins. Also, we decided to remove the plot showing the scaling of the antiflatness, since the precise scaling behavior should be carefully extracted from that of the Rényi entropies. However, we chose to retain in the text the statement that this quantity is expected to exhibit a power-law behavior, as this remains true regardless of the exact coefficients appearing in the scaling of the 2-Rényi and 3-Rényi entropies (see lines 635-636).

References

[1] Fabio Franchini. *An Introduction to Integrable Techniques for One-Dimensional Quantum Systems*. Springer International Publishing, 2017.

[2] Salvatore M. Giampaolo, Gerardo Adesso, and Fabrizio Illuminati. Separability and ground-state factorization in quantum spin systems. *Phys. Rev. B*, 79:224434, Jun 2009.