Reply to Referee B

We thank Referee B for the positive assessment of our work and for the valuable comments and questions, which have helped us improve the manuscript. Our responses to these points are given below.

Referee B: *1(a). I believe the manuscript lacks a discussion of the physical meaning of the proposed quantities. As stated after Eq.(1) for closed systems the "LE measures the sensitivity of quantum evolution to the perturbation and quantifies the degree of irreversibility". What about open systems?*

Reply: We thank the referee for this very insightful question. In closed systems, the LE quantifies the sensitivity of unitary time evolution to a Hamiltonian perturbation and thereby characterizes the degree of irreversibility. In contrast, for open systems, the dynamics is no longer unitary, and the notion of irreversibility must take into account not only Hamiltonian perturbations but also changes in the dissipative processes.
In the open-system setting, the dynamics is generated by two Lindbladians $\mathcal{L}_1$ and $\mathcal{L}_2$, starting from the same initial state ρ_0, so that $\rho_\alpha(t) = e^{t\mathcal{L}_\alpha}(\rho_0)$ for $\alpha = 1, 2$. The open-system LE that we propose,

$$M^D(t) \equiv \frac{\mathrm{Tr}[\rho_1(t)\rho_2(t)]}{\sqrt{\mathrm{Tr}\big[\rho_1^2(t)\big]\ \mathrm{Tr}\big[\rho_2^2(t)\big]}} = \frac{\langle\psi_0^D|e^{iH_2^{D\dagger}t}e^{-iH_1^D t}|\psi_0^D\rangle}{\sqrt{\langle\psi_0^D|e^{iH_1^{D\dagger}t}e^{-iH_1^D t}|\psi_0^D\rangle\,\langle\psi_0^D|e^{iH_2^{D\dagger}t}e^{-iH_2^D t}|\psi_0^D\rangle}},$$

is the normalized Hilbert–Schmidt overlap between the two time-evolved density matrices. Thus, the open-system LE quantifies how distinguishable the two Lindbladian evolutions are, and therefore measures the sensitivity of the open-system dynamics to perturbations of the full Lindbladian. These perturbations may arise from changes in the Hamiltonian, variations in the strength of dissipation, or modifications of the jump operators.
In this way, the open-system LE provides a natural generalization of the closed-system LE: it characterizes irreversibility and sensitivity not only to Hamiltonian perturbations but also to noise-induced and dissipative perturbations inherent to open-system dynamics.
We have also included this explanation in the main text at section 2 under the Eq(7).

Referee B: *1(b). It seems to me that the physics is slightly different: closed systems do not posses a global stationary state (only reduced density matrices may have one) while in this case there is one and its dependence on the perturbation distinguishing L_1 and L_2 determines the asymptotic value of $M^D(t)$. If it is one than there is no sensitivity of the stationary state to a change of system parameters. How should one interpret the intermediate dynamics?*

Reply: We thank the referee for this very thoughtful question. We fully agree that, in contrast to closed systems, generic open systems possess a stationary state and that its dependence on the perturbation distinguishing $\mathcal{L}_1$ and $\mathcal{L}_2$ controls the asymptotic value of $M^D(t)$. However, the intermediate-time dynamics probed by the LE is not fixed by the stationary state alone, but by the full Lindblad spectrum. In particular, the time scales and structures that we analyze (locations of minima, crossover scales, etc.) are governed by the eigenvalues and eigenmodes of the Liouvillian, which depend on both the Hamiltonian and the dissipative part. Thus, even when the stationary state is the same for $\mathcal{L}_1$ and $\mathcal{L}_2$, the transient dynamics encoded in $M^D(t)$ is highly nontrivial and remains sensitive to the perturbation.

In the SYK models we focus on in the manuscript, the Lindblad jump operators are Hermitian and the dynamics is ergodic. In this situation the unique stationary state is the infinite-temperature density matrix, proportional to the identity, and it is independent of both the Hamiltonian parameters and the dissipation strength. We chose this setting for two main reasons.

First, this situation is quite common in physically relevant open quantum systems. Local Hermitian jump operators, such as dephasing or certain classes of spin-relaxation channels, typically drive the system toward an infinite-temperature steady state. In such cases, both evolutions generated by $\mathcal{L}_1$ and $\mathcal{L}_2$ relax to the same stationary state, so that $M^D(t \to \infty) = 1$. Although the stationary state itself is then insensitive to the perturbation, the intermediate-time dynamics is not: $M^D(t)$ still quantifies how differently the two Lindbladians process the same initial state on their way to the common fixed point. The minima and time scales that we analyze are precisely measures of this transient dynamical sensitivity, which is controlled by the Liouvillian spectrum rather than by the steady state alone.

Second, if one considers more general Lindbladians for which the stationary state depends on the Hamiltonian parameters and on the dissipation strength, then the two evolutions will in general relax to different stationary states ρ_1^{ss} and ρ_2^{ss}, and the long-time value of the LE will be a plateau strictly smaller than one, determined by their (normalized) overlap. In that more general case, however, the full time dependence of $M^D(t)$ becomes strongly model dependent, and we are not aware of any universal statements that can be made beyond this asymptotic overlap. Since our goal here is to identify robust and universal structures in the LE dynamics, we have therefore concentrated on the broad and experimentally relevant class of dynamics with a common, parameter-independent infinite-temperature stationary state, where we can make controlled and model-independent statements.

To demonstrate that our results are not peculiar to the SYK model, we have included in the revised manuscript an explicit example based on the XXZ spin chain,

$$H = J \sum_j \left(S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + \Delta S_j^z S_{j+1}^z \right),$$

with local dephasing described by jump operators $L_j = S_j^z$ on each site. The corresponding Lindblad evolution,

$$\frac{\partial \rho}{\partial t} = -i[H, \rho] + 2\gamma \sum_j S_j^z \rho S_j^z - \gamma \sum_j \left\{ (S_j^z)^2, \rho \right\},$$

again has the infinite-temperature state as its unique stationary state. This XXZ model, unlike the all-to-all disordered SYK model, has only nearest-neighbor interactions and is widely realized in platforms such as ultracold atoms in optical lattices and superconducting-qubit architectures. Our numerical simulations show that its LE exhibits the same qualitative behavior as in the SYK case: a single minimum in the weak-dissipation regime and a clear two–local-minima structure in the strong-dissipation regime, robust across gapped ferromagnetic, gapless Luttinger-liquid, and gapped antiferromagnetic phases (see Appendix C).

Thus, the LE behavior we uncover is neither specific to SYK nor trivialized by the presence of a parameter-independent stationary state. Rather, it reflects generic dynamical features of open quantum systems with a common infinite-temperature steady state, governed by the Liouvillian spectrum. In the weak-dissipation regime, the single minimum follows from a perturbative analysis that does not rely on any special property of the SYK Hamiltonian, while in the strong-dissipation regime, the emergence of two local minima can be traced to the spectral structure of the doubled-space generator under conditions that are again not unique to SYK.

We have added the XXZ simulations and a discussion of these points to Appendix C. We have also revised the main text (adding a paragraph at the end of Sec. 3, updating the introduction, and adding a second paragraph to the conclusions) to clarify that, although our explicit calculations are performed in the SYK and XXZ models, the underlying mechanism is general.

Referee B: *1.(c) Same question for the OTOC: the physical meaning of the OTOC for closed systems is connected to the square commutator and its semiclassical representation suggesting that its exponential growth will detect many body chaos. What kind of information would one get from the proposed quantity? What motivates physically its usefulness for open systems?*

Reply: We thank the referee for this very insightful comment. The definition of the OTOC in open systems that we use in the main text is a direct generalization of the standard closed-system OTOC. It is given by

$$F^D(t) = \frac{1}{d}\mathrm{Tr}\left\{R_B^\dagger\, e^{\mathcal{L}^\dagger t}\big[W_A^\dagger\, e^{\mathcal{L}t}[R_B]\, W_A\big]\right\},$$

where $\mathcal{L}$ is the Lindblad generator. When dissipation is switched off, and $\mathcal{L}$ reduces to a purely Hamiltonian Liouvillian, this expression collapses to the usual OTOC for closed systems. Moreover, this form is particularly well suited to existing NMR protocols, where time reversal of the effective dynamics can be implemented rather directly; we discuss this point in more detail in Sec. 6 of the manuscript.

Physically, the open-system OTOC $F^D(t)$ probes the competition effects of unitary information scrambling and environmental decoherence. For a chaotic Hamiltonian subject to dissipation, its decay at early times is controlled both by the scrambling dynamics and by the dissipative channels, and thus encodes their competition. A systematic study of how this decay can be decomposed into "chaotic" and "dissipative" contributions, and of its detailed scaling properties, would be a very interesting problem in its own right.

A quantitative study of this question, however, lies beyond the scope of the present work. Here, our focus is on (i) introducing a consistent definition of the OTOC for open systems, (ii) establishing its relation to the Loschmidt echo in the Lindbladian setting, and (iii) outlining a concrete measurement protocol. A more detailed investigation of the competition between scrambling and dissipation is an appealing direction for future research. We have also added the following remark to the main text at section 4.1 under Eq.(24) to clarify the physical meaning of the OTOC in open systems.

Referee B:

2. Are the generalisations of the LE and OTOC to open systems unique? A discussion of this point would be useful.

Reply:

We thank the referee for this very helpful question. Our proposed generalizations of the LE and OTOC to open quantum systems are indeed *not* unique. In the presence of dissipation, there is no single canonical definition of either quantity. Our work, therefore, adopts one specific generalization that is natural for the experimental setup considered in Section 6.

In our paper, the OTOC for an open system is defined as

$$F^D(t) = \frac{1}{d}\,\mathrm{Tr}\Big\{R_B^\dagger\, e^{\mathcal{L}^\dagger t}\Big[W_A^\dagger\, e^{\mathcal{L}t}[R_B]\, W_A\Big]\Big\}\,.$$

We use this definition because it admits a direct and natural implementation in NMR experiments, as discussed in Sec. 6 of the main text. In particular, it is tailored to the specific measurement protocol that accesses the reduced dynamics of an operator R_B conditioned on manipulations on subsystem A.

Related definitions and interpretations of OTOCs in open quantum systems have been discussed in the literature. For example, Ref. [Phys. Rev. B **104**, 035425 (2021)] considers

$$F_{\mathcal{E}}(t) = \frac{1}{d}\,\mathrm{Tr}\Big\{\mathcal{E}(R_B^\dagger)\, W_A^\dagger\, \mathcal{E}(R_B)\, W_A\Big\}\,,$$

where $\mathcal{E}(\cdot)$ is a quantum channel and $\mathcal{E}$ denotes the adjoint (Heisenberg-picture) action on operators. In general, the above definition is not identical to our definition. However, when the time evolution of the open system is generated by a Lindblad equation with Hermitian jump operators, and when one specializes to the same bipartite setting and operator choices, the two definitions coincide.

Another prominent example is the *averaged* OTOC for a channel $\mathcal{E}$, defined as (see, e.g., Phys. Rev. A **103**, 062214 (2021))

$$G(\mathcal{E}) = \frac{1}{2d}\,\mathbb{E}_{V_A,W_B}\big\|[\mathcal{E}(V_A), W_B]\big\|^2,$$

where $\mathcal{E}(\cdot)$ is a general quantum channel, V_A and W_B are local operators on subsystems A and B respectively, and the expectation is over a suitable ensemble of local unitaries. This bipartite open-system OTOC can also be interpreted as a notion of distance between channels and as a probe of entropy production. Here the focus is on out-of-time-ordered *commutators*, whereas our quantity $F^D(t)$ is an out-of-time-ordered *correlator* for a fixed pair of operators (W_A, R_B). When the channel is generated by a Lindbladian with Hermitian jump operators, and when one averages our correlator-based OTOC over all local operators on A and B in analogy with $G(\mathcal{E})$, the resulting averaged correlator becomes equivalent, in this specific sense, to the commutator-based measure. In general, however, these definitions capture different aspects of scrambling and dissipation and are not strictly identical.

The LE in open systems is likewise not unique. In a reduced-dynamics language, another definition (see, for example, Phil. Trans. R. Soc. A **374**, 20150162 (2016); arXiv:2104.06367; Opt. Commun. **372**, 158–163 (2016)) is as follows. If $\Lambda_0(t)$ and $\Lambda_1(t)$ are two CPTP maps on the open system induced by different total evolutions U_0 and U_1, one defines

$$M_{\text{open}}(t) = \left|\text{Tr}\big[\rho_S(0)\,\Lambda_1^\dagger(t) \circ \Lambda_0(t)\big(\rho_S(0)\big)\big]\right|^2,$$

where $\rho_S(0)$ is the initial state of the system. In this definition, there is no explicit normalization factor: the decay of $M_{\text{open}}(t)$ combines the effects of dissipation (loss of purity) and the sensitivity to the perturbation distinguishing Λ_0 from Λ_1.

By contrast, in our work we use the normalized LE

$$M^D(t) \equiv \frac{\text{Tr}[\rho_1(t)\rho_2(t)]}{\sqrt{\text{Tr}\big[\rho_1^2(t)\big]\ \text{Tr}\big[\rho_2^2(t)\big]}} = \frac{\langle\psi_0^D|e^{iH_2^{D\dagger}t}e^{-iH_1^D t}|\psi_0^D\rangle}{\sqrt{\langle\psi_0^D|e^{iH_1^{D\dagger}t}e^{-iH_1^D t}|\psi_0^D\rangle\,\langle\psi_0^D|e^{iH_2^{D\dagger}t}e^{-iH_2^D t}|\psi_0^D\rangle}}\,,$$

where $\rho_{1,2}(t)$ are the two reduced states generated by different Liouvillians and $|\psi_0^D\rangle$ is the corresponding doubled-state purification. The normalization in the above equation removes the trivial contribution from the decay of purity under each individual open evolution. As a result, $M^D(t)$ more cleanly isolates the sensitivity to the *difference* between the two Lindbladians, rather than conflating this with the overall strength of dissipation.

Since both LE and OTOC admit several reasonable generalizations in open systems, the relation we derive is not unique and relies on two main ingredients. First, we average over operators W_A acting on subsystem A in the OTOC, which allows us to rewrite the OTOC in terms of the reduced time evolution of an operator R_B on subsystem B; the resulting LE–OTOC relation is therefore tied to this averaging procedure. Second, we model the reduced dynamics of R_B using a random-noise description, in which R_B evolves under an effective Hamiltonian on B plus stochastic noise determined by the coupling between A and B. The LE–OTOC connection we obtain is thus valid within this specific, but physically motivated, framework.

A more direct and general relation between LE and OTOC in open quantum systems—one that avoids both the averaging over random operators and the random-noise approximation—remains an open problem. In particular, without such averaging it is not obvious how to extract the asymmetric "forward vs. backward" structure of the LE directly from an OTOC. Exploring alternative definitions and more direct LE–OTOC relations in open systems is an interesting direction for future work. We now add a brief comment on this point in the final paragraph of Sec. 4 in the main text.

Referee B: *3. In view of point 1 of this section the example chosen is quite peculiar: the stationary state is insensitive to the system parameters, hence the LE always tends to one. Since at t=0 starts at one, it cannot do much but have one or more local minima which are described in the manuscript but whose physical meaning is not discussed. What happens if one considers a more interesting situation in which the stationary state does depend on the system parameters and on dissipation?*

Reply: We thank the referee for this very insightful question. Indeed, in the example considered in our paper, the Lindbladian has a unique stationary state proportional to the identity. As a consequence, the long-time plateau of the open-system LE we propose is always equal to one, since the evolutions with dissipation strengths γ_1 and γ_2 relax to the same steady state. However, even though the steady state is the same, the dynamics is not trivial. As we discuss in the main text, the open-system LE displays nontrivial and, in fact, *universal* structures in time: the overall scaling and the positions of the minima are controlled by the Lindblad spectrum, which depends on both the Hamiltonian parameters and the dissipation strength. In particular, the emergence of a single minimum in the weak-dissipation regime and of two local minima in the strong-dissipation regime reflects a segmented structure of the Lindblad spectrum.
If instead the stationary state of the open evolution depends on the system parameters and on the dissipation strength, then in general the steady states of the γ_1 and γ_2 evolutions will be different. In that case, the long-time value of the LE would approach a plateau smaller than one, determined by the overlap of the two stationary states,

$$M^D(t \to \infty) = \frac{\mathrm{Tr}(\rho_1 \rho_2)}{\sqrt{\mathrm{Tr}(\rho_1^2)\,\mathrm{Tr}(\rho_2^2)}}\,,$$

where ρ_1 and ρ_2 are the respective stationary states. When the stationary states depend on the Hamiltonian parameters and on the dissipation, the full LE dynamics becomes strongly model dependent. In such a situation we do not expect universal, model-independent features of the kind we identify in the identity-steady-state case: for example, the LE can either decrease or increase in a non-monotonic fashion, and its detailed behavior is governed by system-specific properties of the stationary states and of the corresponding Liouvillian spectrum.
By contrast, in the present work, we have deliberately focused on the simple but widely relevant case in which the two evolutions share the same stationary state (the infinite-temperature state in our example). This scenario is not at all exceptional: it arises quite naturally when dissipation is generated by Hermitian jump operators, for which the unique stationary state is often proportional to the identity. In this setting, we can make robust and largely model-independent statements about the full time dependence of the LE, such as the appearance of a single minimum in the weak-dissipation regime and two local minima in the strong-dissipation regime, which we interpret in terms of the Lindblad spectrum.

We fully agree that situations with nontrivial stationary states depending on the Hamiltonian parameters and on dissipation can exhibit qualitatively different dynamical behavior and interesting physics (for example, when the steady state corresponds to different symmetry-broken phases or distinct steady-state phases). A systematic exploration of these more complex cases would, however, require a separate and more detailed analysis and is beyond the scope of the present work. We therefore leave a thorough investigation of LE dynamics in such nontrivial steady-state scenarios as an interesting direction for future research.