Reply to Referee comments on

Hydrodynamics without Averaging – a Hard Rods Study

Dear editor and referees,

thank you very much for your time and effort to review my paper, for the positive reviews and for the useful comments that I have used to improve the manuscript.

Beneath you find my response to each referee individually.

I have updated the manuscript taking into account the comments of the referees. To identify them easily, I colored the major changes are colored in orange (I did not color minor changes like fixing typos etc). One comment raised by most of the referees is that it was not clear what is the starting point, the goal and the conclusion of each computation. I agree that my previous presentation was lacking there. I have added more explanation to all of the sections, making it clear in each section what quantities are being computed and also how to interpret the individual results.

Kind regards,
Friedrich Huebner

PS: I fixed the arxiv references.

1 Referee 1

- *In general, it would be good to add an additional section "main results", explaining and summarizing the results of each section* Good idea. I have added such a section.

- *It will be also good to announce the results and explain the logic of derivation at the beginning of each section. And more importantly trying to exactly pronounce the starting point: equations and assumptions from which the rest is derived. For instance:I assume that the evolution of the coarse-grained system is further captured by Eq. 8, however it was never explicitly stated. It would be good to comment on that directly when writing down equations 76-79 and also 119-120.* I agree that the presentation in these sections was incomplete. I have added further details about the starting point, the goal and the conclusion of each computation.

- *It will be good to discuss what are the author's expectations in case of a general integrable systems. Does the introduced approach allow to derive (or at least guess) the generalization of Eq. 159 to the case of a generic integrable system. If not, briefly comment why.* I have added remark 10 after the equation: Yes, I expect that a very similar equation will hold in general integrable models. This was proposed by me and my collaborators in [1]. Unfortunately, it is not possible to do explicit microscopic computations like for hard rods. Nonetheless, the equation can still be derived by assuming that there is no intrinsic diffusion and all diffusion arises from "diffusion by convection". The outcome reduces to the correct result for the independently studied special case of local equilibrium states [2, 3] (where long range correlations are absent). This is a strong hint, that intrinsic diffusion is indeed absent in integrable systems in general.

- *The definition of $z_{\alpha,\beta,\alpha',\beta'}$ between 85 and 86 is inaccurate or deserves explanation, because the RHS of the definition does not contain α', β', containing instead $\alpha(x_j), \beta(x_j)$.* There was a typo, the definition of $z_{\alpha,\beta,\alpha',\beta'}$ should certainly not depend on j. The correct expression is $z_{\alpha,\beta,\alpha',\beta'} = \hat{x}_\alpha + p_\beta t - \hat{x}_{\alpha'} - p_{\beta'} t$. I have corrected it.

- *The estimation for the number of summation terms in Eq. 87 is not* *clear from the text, in particular what are the summation bounds. It is also not clear why $z_{\alpha,\beta,\alpha',\beta'} = \mathcal{O}(\Delta x)$, naively it looks that for different cells α, α' the difference $\hat{x}_\alpha - \hat{x}_{\alpha'}$ might be of order one, which is $O(1)$.* For generic $\alpha, \beta, \alpha', \beta'$, $z_{\alpha,\beta,\alpha',\beta'}$ is indeed of $\mathcal{O}(1)$. Also, the sum in (95, new version) runs over all $\alpha, \beta, \alpha', \beta' \in \mathbb{Z}$. However, since $y_{ij}(t) \sim \mathcal{O}(\Delta x)$, the indicator function $\theta(0 < -\operatorname{sgn}(y_{ij}(t)) z_{\alpha,\beta,\alpha',\beta'} < |y_{ij}(t)|)$ is only non-zero iff $z_{\alpha,\beta,\alpha',\beta'} = \mathcal{O}(\Delta x)$, which significantly restricts the sum over $\alpha, \beta, \alpha', \beta'$. For instance, fixing α, β, β', there will only be very few (typically one or two) α' such that $z_{\alpha,\beta,\alpha',\beta'} = \mathcal{O}(\Delta x)$.

- *The equations 159-160 deserve better explanation, in particular, it seems that they should be complemented by an equation for $G_{LR,sym}$ to be complete. Without such a relation, it is not clear how to use 159-160 alone for the numerical simulation.* I agree, the equation for the correlation functions should be there. I have added it now, see equation (175).

- *As a minor suggestion: it will be good to use multline envitonment for a multiline equuations like 159-160 (and many others). Otherwise both lines get enumerated, which is not common.* I agree. I made sure that each equation is labeled only once, even if it goes over multiple lines. The only exception to this is (62) and (63), where I refer to both lines separately in the text beneath.

2 Referee 2

- *Minor typos (page 4 'jugding' instead of judging)* Fixed.
- *There is a missing Delta x in Eq. 26* Fixed.
- *In figure 3 the blank plot should be removed.* I would prefer to keep it there. Even though it does not show numerical results, it still shows the theoretically obtained exponents. However, I understand that it looks empty. I have altered the plot slightly, emphasizing the theoretical result.
- *Under remark 9 the sentence: "If on the other hand, one is able to only measure observables averaged over many initial states, then one should use (160) instead." Is a bit unclear to me. I take it to mean that (160) captures the dynamics that emerge having averaged over the initial states. Perhaps this could be slightly reworded to clarify the difference between (8) (evolve many samples then average) vs (160) (average initial states then evolve).* Yes, you interpret the sentence correctly. I have reworked the paragraph to make it more clear. Is it clearer now?
- *The author points out Ginibre states as being particularly unphysical thus demonstrating the robustness of GHD. Does the author have any insight about the types of ensemble of states that are not captured by GHD?* This is an interesting question. In general in hydrodynamic limits/mean field limits one expects that there are pathological configurations that give rise to all sorts of unusual behaviour[1]. Typically these are "measure zero sets" (so unphysical). In the hard rods, the question is whether the continuum limit of (5-7) can be taken to obtain (11-12). For sufficiently well behaved states this should always be possible. Hence, for all intended purposes I would say that GHD always applies. However, I imagine GHD cannot be applied in the following pathological scenarios
  - The initial density $\rho(x,p)$ does decay very slowly for x or $p \to \pm\infty$, so that the continuum limit can not be taken. Also, in case of very

[1] A typical situation is a configuration where 3 hard spheres in $d > 1$ dimension meet simultaneously. At this point the evolution cannot be defined uniquely.

slow decay in p, the nominator of the effective velocity (10) is not defined.

- The initial configuration is sparse, say the initial $x_j = 2^j$, so that in the continuum limit the density everywhere is formally vanishing.
- Initial particles are very precisely located inside fluid cells.
- Many particles cluster very close together (such that the denominator of the effective velocity (10) explodes).
- A similar effect (which however does not appear in hard rods), is a macroscopic scattering event, where a macroscopic number of particles scatter for a macroscopic time, leading to a breakdown of GHD for that time in the scattering region. In case you are interested, this is explained in my PhD thesis on the example of negative length hard rods [4], see in particular figure 5.2.

3 Referee 3

- *The presentation of the derivation in sections 3 and 4 can be improved. If possible, it would be useful to clearly mention the motivation, goal and the plan of the derivation at the beginning of each section+ subsection. Currently the conclusion of the sections are also not very clearly stated.* I agree that the presentation in these sections was incomplete. I have added further details about the starting point, the goal and the conclusion of each computation.

- *In BMFT it is typically assumed that the for coarse-gaining scale 'cg $\sim$ 'variation and in such scale also the evolution of the initial fluctuations are described by Euler equations. For coarse graining size $\mu = 1/2$ or $> 1/2$, it seems there will be significant noise. Does it imply one needs to look at such noise to understand (corrections to) correlation at diffusive space-time scale?* My understanding of the large fluid cell case $\mu > 1/2$ is that the error is dominated by a systematic error (as opposed to a statistical error for $\mu < 1/2$). The fluid cells are so big that one does not have sufficient information to predict the value of an observable more precisely, even if statistical fluctuations were absent. This already appears after initial coarse-graining, so I believe that this is a general coarse-graining effect, independent of the time evolution. Also, one can always choose smaller $\mu < 1/2$ to obtain a lower error.

  About BMFT: Note that (Euler) BMFT only requires that for each sample Euler GHD is correct up to an error that decays as $\ell \to \infty$. For $\mu > 1/2$ the error decays as $\ell^{-2(1-\mu)}$, meaning that (Euler) BMFT also can be used with $\mu > 1/2$. What is special about $\mu < 1/2$ is that the error is beneath $1/\ell$, meaning that BMFT is exact also on the diffusive scale (one might call it diffusive BMFT). So no, I do not think that any further noise has to be taken into account to study correlation functions on the diffusive scale.

- *The proof of Eq. (137) is not given in Appendix A* I agree, I forgot to put it. It follows from the explicit formula (11). I rewritten that paragraph.

- *It would be good to write the representation of the operator $\hat{D}$ in Eq. (156).*I agree, I have put it there.

- *Eq.(A1) and (A2) are same and (A2) has a typo.* I agree. I have removed (A2).
- *(vi) Eq. (A12): beginning of the second last line:* $+2a \to 2a^3$I agree. Fixed it.
- *Eq. (A27): $A(x, p)$ in the second term on the right.*I agree. Fixed it.
- *First line of sec. (A.2): the correlation should be CLR according to the notation in Eq. (131).* I agree that this is confusing. I forgot to say that in this paragraph I consider only the long range correlations, i.e. as if there was no singular part. This splitting is allowed since the correction to the current is linear in the correlations, so one can study the singular and long range part separately. I have improved the paragraph.

4 Referee 4

- *The introduction is very long, and not very focused. It could be more concise, and 'to-the-point'. A summary of the results would be appreciated. Also , in the first paragraph of the introduction the author seems to be talking about integrable systems since he is talking about generalized Gibbs ensembles. But then the later discussion in the introduction seems to be aimed at all systems, including non-integrable. Please clarify whether this introduction is about integrable or generic systems.*

  I agree that the introduction is very long. The reason for this is that I wanted to embed the work in the context in hydrodynamics, and especially the open problems therein: I wanted to highlight that the advantage of the "hydrodynamics without averaging" approach to better understand the physics beyond the Euler scale or of correlation functions, as compared to more traditional approaches. For now, I did not change the introduction, but if you have a strong opinion that it would improve the paper, then I am also open to make it more concise.

  However, in combination with a comment of referee 1, I added a new section 3 summarizing the main results of the paper.

  About integrable vs non-integrable: The introduction considers a general system, integrable or non-integrable. While GGE is the common terminology for integrable systems, but the concept is valid more general. In a system with N (finite or infinite) conservation laws Q_n, the GGE is a state of the form $\frac{1}{Z}e^{-\sum_{n=1}^{N}\beta_n Q_n}$. In the context of a usual Galilei invariant system (conserving particle number, momentum and energy) these reduce to usual Gibbs states. Thus the introduction is about a general system (until I start to discuss hard rods). I have changed to paragraph to make it more clear.

- *Sections 3.1 ,3.2,3.3,3.4 and 5.2 merely look like technical notes. Please provide a roadmap to the reader at the beginning of each subsection to explain what is the goal of the calculations* I agree that the presentation in these sections was incomplete. I have added further details about the starting point, the goal and the conclusion of each computation.

- *Please clarify whether / which conclusions of this paper hold beyond the integrable case.* I think this is an important point which I did not emphasize in the conclusion. I have added an extra paragraph in the conclusion, see section 8.1.

- *'Weak solution' in page 7 does not seem to be defined. Please define.* Good point. I have added the definition.

- *There are many typos (p.2 'would be differ', p.4 'hards', p.8 'is that is that', 'corase-graining', etc). Please proofread.* Thank you for spotting these. I have corrected them and some further typos.

References

[1] Friedrich Hübner et al. "Diffusive Hydrodynamics from Long-Range Correlations". In: *Phys. Rev. Lett.* 134 (18 2025), p. 187101. DOI: 10.1103/PhysRevLett.134.187101. URL: https://link.aps.org/doi/10.1103/PhysRevLett.134.187101.

[2] Jacopo De Nardis, Denis Bernard, and Benjamin Doyon. "Diffusion in generalized hydrodynamics and quasiparticle scattering". In: *SciPost Phys.* 6 (2019), p. 049. DOI: 10.21468/SciPostPhys.6.4.049. URL: https://scipost.org/10.21468/SciPostPhys.6.4.049.

[3] Jacopo De Nardis, Denis Bernard, and Benjamin Doyon. "Hydrodynamic Diffusion in Integrable Systems". In: *Phys. Rev. Lett.* 121 (16 2018), p. 160603. DOI: 10.1103/PhysRevLett.121.160603. URL: https://link.aps.org/doi/10.1103/PhysRevLett.121.160603.

[4] Friedrich Hübner. "On the Hydrodynamic Approximation of Quantum Integrable Models: An Illustration via the repulsive Lieb-Liniger Model". PhD thesis. King's College London, 2025. URL: https://kclpure.kcl.ac.uk/portal/en/studentTheses/on-the-hydrodynamic-approximation-of-quantum-integrable-models/.