Bootstrapping the R-matrix

Zhao Zhang[1*]

1 Department of Physics, University of Oslo, P.O. Box 1048 Blindern, Oslo, N-0316, Norway

⋆ zhao.zhang@fys.uio.no

Abstract

A bootstrap program is presented for algebraically solving the R-matrix of a generic integrable quantum spin chain from its Hamiltonian. The Yang-Baxter equation contains an infinite number of seemingly independent constraints on the operator valued coefficients in the expansion of the R-matrices with respect to their spectral parameters, with the lowest order one being the Reshetikhin condition. These coefficients can be solved iteratively in a self consistent way using a lemma due to Kennedy, which reconstructs the R-matrix after an infinite number of steps. For a generic Hamiltonian, the procedure could fail at any step, making the conditions useful as an integrability test. However in all known examples they all turn out to be satisfied whenever the lowest order condition is. It remains to be understood whether they are all implied by the Reshetikhin condition.

Contents

1   Introduction

In statistical mechanics courses, the R-matrix is usually introduced in vertex models as a way to compactly encode the Boltzmann weights, which can be used to conveniently compute the partition function with the transfer matrix method. The success of this calculation depends on the commutativity between column transfer matrices, which is guaranteed by the Yang-Baxter equation (YBE) satisfied by integrable models. For every two-dimensional (2D) integrable statistical mechanical model, one can derive the quantum Hamiltonian for a related one-dimensional integrable spin chain from the transfer matrix. However, it is not always clear how to find the 2D statistical mechanical dual model from the Hamiltonian of a quantum spin chain.

Nevertheless, without an R-matrix that can be used to test integrability with the YBE, the integrability of quantum spin chains has been successfully tested by the Reshetikhin condition [1], which involves only the local Hamiltonian terms. It has hence been conjectured to be both a sufficient and a necessary condition for quantum integrability [2]. In practice, the Reshetikhin condition has been tested using Kennedy's inversion formula [3], which is actually also the first step towards constructing the R-matrix from an integrable Hamiltonian that has so far been overlooked.

To fully unleash the power of Kennedy's method, one can try Taylor expanding a fictitious R-matrix with respect to its spectral parameter demanding the first order term to be proportional to the Hamiltonian in question. This has been explored in Ref. [4,5], in order to obtain higher order integrability tests than Reshetikhin's condition. However, their attempts were not so successful, for two good reasons. Firstly, they did not use the identities implied by the lower order terms in the YBE to simply the higher order ones to a symmetric form that resembles the Reshetikhin condition. Secondly and more importantly, they have not considered the fact that constant shifts to the Hamiltonian should not matter for its integrability. This is the reason that their expansions with the zeroth order term having coefficient 1 would not work for certain R-matrix with the coefficient for the identity operator being a function of the spectral parameter, such as the Takhtajan-Babujian spin-1 model [6,7]. As a result, these Hamiltonians, despite integrable, would violate the second order integrability condition they obtained.

In this article, I take into account both considerations to construct a bootstrap program for iteratively solving the R-matrix corresponding to any integrable local Hamiltonian as long as its R-matrix depends only on one spectral parameter. The procedure is new in that the computations are purely algebraic, without having to solve any differential equations. Therefore it can be implemented in symbolic calculations at a very affordable cost even for large local degrees of freedom. At each step of the program, a higher order version of the Reshetikhin condition is used, which are also apparently independent integrability conditions. Together, they imply the conservation of higher order charges that commute with the Hamiltonian. In addition, I have also generalized Kennedy's inversion formula to operators with larger supports that allows the computation of explicit forms of the charge densities. The generalization itself reveals a discrete conformal algebra structure, which together with an analysis of the lattice

Poincaré group strengthens the understanding of higher order charges as the same conserved charge observed in different reference frames.

The rest of the article is organized as follows. Sec. 2 reviews the Reshetikhin condition as derived from Taylor expanding the R-matrix with respect to spectral parameter, and its relation to the conservation of the three-local charge. Sec. 3 derives higher order forms of the integrability conditions, highlighting the apparent independence among them, and their usage as integrability test for a generic Hamiltonian, and iterative construction of the R-matrix for integrable Hamiltonians. Sec. 4 parametrizes generic bi-local Hamiltonians with $\mathfrak{sl}(n, \mathbb{C})$ matrices to illustrate how the condition can be transformed into equations involving the coupling constants of the interaction. Sec. 5 introduces the boost operator that generates higher conserved charges from lower ones, and the associated time dependent symmetry manifest in the Heisenberg picture. Sec. 6 takes an interlude of the lattice Poincaré group to give an intuition of how higher conserved charges generated by the boost operator could possibly be the same three-local charge observed in different reference frames. The heuristic picture gained in Sec. 6 is then connected back to the theme of conserved charges by the introduction of a discrete conformal algebra in Sec. 7. The difficulties of generalizing the bootstrap program to non-relativistic integrable Hamiltonians with R-matrices that depends on two spectral parameters are explained in Sec. 8. Finally, the article concludes by discussing a few possible future directions in Sec. 9.

2 Reshetikhin's condition and the continuity equation

A quantum spin chain with local interactions of range r can be mapped into another chain with blocks of size r, such that the new system has an enlarged local Hilbert space but interacts only between nearest neighbors. Therefore, it suffices to consider Hamiltonians of the form $H = \sum_x h_{x,x+1}$, where the 2-local operator $h_{x,x+1} = \cdots \otimes 1_{x-1} \otimes h_{x,x+1} \otimes 1_{x+2} \otimes \cdots$ acts nontrivially only on the tensor product of local Hilbert space at site x and $x+1$. The range of the sum in the definition of the Hamiltonian has been deliberately omitted by assuming the chain to be infinitely long, so that we do not have to worry about boundary conditions. By definition of the local Hamiltonian densities $h_{x,x+1}$, it is apparent that they commute with each other unless their supports overlap: $[h_{x,x+1}, h_{x',x'+1}] = 0$ if $x' \neq x-1, x+1$.

For integrable Hamiltonians with R-matrices that depend only on the the difference of two rapidities, a hypothetical R-matrix that depends on only one spectral parameter can be constructed from the Hamiltonian density $h_{x,x+1}$,

$$\check{R}_{x,x+1}(\xi) = 1_{x,x+1} + \sum_{n=1}^{\infty} \frac{\xi^n}{n!} \check{R}_{x,x+1}^{(n)}, \tag{1}$$

where $\check{R}_{x,x+1}^{(1)} = h_{x,x+1} + c 1_{x,x+1}$. The constant shift c in the linear term accounts for different normalizations of the R-matrix. Since the higher order operators $\check{R}_{x,x+1}^{(n)}$ are yet to be determined, by including a term proportional to identity in the first order, all those integrable Hamiltonians can also be included in the picture. The flexibility that comes with the constant linear term also explains why the arbitrariness of the Hamiltonian of shifting by a constant should not alter whether the system is integrable or not: as long as there is a choice of c that makes (1) satisfy the YBE, the Hamiltonian is integrable.

The R-matrix so constructed from a relativistic integrable Hamiltonian satisfies the braided form of the YBE

$$\check{R}_{x,x+1}(\zeta)\check{R}_{x-1,x}(\xi)\check{R}_{x,x+1}(\xi-\zeta) = \check{R}_{x-1,x}(\xi-\zeta)\check{R}_{x,x+1}(\xi)\check{R}_{x-1,x}(\zeta). \tag{2}$$

Taking $\xi = 0$, it implies the unitarity condition

$$\check{R}_{x,x+1}(\zeta)\check{R}_{x,x+1}(-\zeta) = 1_{x,x+1}. \tag{3}$$

At even powers of ζ, (3) fixes the even order operators in (1) in terms of lower order ones

$$\check{R}_{x,x+1}^{(2m)} = \frac{1}{2}\sum_{k=1}^{2m-1}(-1)^{k-1}\binom{2m}{k}\check{R}_{x,x+1}^{(k)}\check{R}_{x,x+1}^{(2m-k)}, \tag{4}$$

which makes $\check{R}_{x,x+1}^{(2)} = (h_{x,x+1} + c1_{x,x+1})^2$.[1] Plugging (1) into (2), and collecting the coefficients of terms proportional to $\xi\zeta^2$ gives Reshetikhin's condition [1]

$$
\begin{aligned}
&[h_{x,x+1},[h_{x,x+1},h_{x-1,x}]]-[h_{x-1,x},[h_{x-1,x},h_{x,x+1}]] \\
&= \left(\check{R}_{x-1,x}^{(3)} - (h_{x-1,x} + c)^3\right) - \left(\check{R}_{x,x+1}^{(3)} - (h_{x,x+1} + c)^3\right).
\end{aligned}
\tag{6}
$$

Up to additional terms that disappear when summed over x, the LHS is the commutator between the Hamiltonian and the charge density $\rho_x \equiv [h_{x,x+1}, h_{x-1,x}]$, while the RHS is a divergence. So (6) is nothing but the continuity equation

$$\partial_t \rho_x = i[H, \rho_x] = j_x - j_{x+1} \tag{7}$$

where the unit $\hbar = 1$ has been adopted. Performing a spatial sum, the RHS cancels telescopically and one recovers the conservation of the three-local charge $Q = \sum_x \rho_x$:

$$[H,Q] = \sum_{x=1}^{L}\left([h_{x-1,x},[h_{x+1,x+2},h_{x,x+1}]] + [H,\rho_x] + [h_{x+1,x+2},[h_{x,x+1},h_{x-1,x}]]\right) = 0, \tag{8}$$

where the first and last term in the summand, accounting for the difference between the LHSs of (6) and (7), cancel due to the Jacobi identity.

3   The R-matrix bootstrap

Higher order generalizations to Reshetikhin's condition (6) and the yet to be defined operator $\check{R}_{j,j+1}^{(2m+1)}$ can be obtained from any of the coefficients of terms homogeneous to $\xi^p\zeta^{2m+1-p}$, for $1 \leq p \leq 2m$. Out of the $2m$ relations, only one is independent. After a manipulation detailed in Appendix A, the identities arising from terms proportional to $\xi\zeta^{2m}$ become

$$
\begin{aligned}
&\frac{1}{2}\sum_{k=1}^{2m-1}(-1)^k\binom{2m}{k}\left(\left[\check{R}_{12}^{(k)},\left[\check{R}_{23}^{(1)},\check{R}_{12}^{(2m-k)}\right]\right] - \left[\check{R}_{23}^{(k)},\left[\check{R}_{12}^{(1)},\check{R}_{23}^{(2m-k)}\right]\right]\right) \\
&= \sum_{k=1}^{2m+1}(-1)^k\binom{2m}{k-1}\left(\check{R}_{12}^{(k)}\check{R}_{12}^{(2m+1-k)} - \check{R}_{23}^{(k)}\check{R}_{23}^{(2m+1-k)}\right).
\end{aligned}
\tag{9}
$$

The fact that an independent condition appears at every second order in the expansion can be compared with the independent charge conservation laws $[H, Q^{(2m+1)}] = 0$ to be discussed in

---

[1]The condition from odd powers of ζ

$$\sum_{k=1}^{2m}(-1)^k\binom{2m+1}{k}\check{R}_{x,x+1}^{(k)}\check{R}_{x,x+1}^{(2m+1-k)} = 0. \tag{5}$$

are always ensured by those from the lower even powers.

the next section (see the right panel of Fig. 1). However, this time it is not straightforward to show the conservation of the corresponding charge as established by (8). Nevertheless, once the R-matrix is constructed order by order, obtaining the conserved charges is simply a matter of expanding the monodromy or column transfer matrix, the commutativity among which at different spectral parameters is guaranteed by the YBE. A few examples of integrable (and partially integrable [8]) spin chains are studied in Appendix B. Specifically for the Heisenberg spin chain, some of these conditions turn out to be generalized forms of the Dolan-Grady relation [9] satisfied by the Onsager algebra [10].

Before detailing this standard procedure in the Sec. 5, I first explain how Eq. (9) can be used to solve $b_{2m-1} \equiv \check{R}^{(2m-1)}_{x,x+1}$ from only the Hamiltonian $b_1 \equiv h_{x,x+1} + c$. Together with (4), this serves as a way to bootstrap the R-matrix $\check{R}(\xi)$ for fundamental integrable Hamiltonians. Kennedy came up with a way to find $\check{R}^{(3)}_{x,x+1}$ from (6) (Lemma 1 in Ref. [3]), which he used to find integrable SU(2) isotropic spin chains. Now, suppose we know already all a_k, b_k for $k \leq 2m$, then the LHS of (9) is completely known, and so is the RHS except a_{2m+1}, b_{2m+1}. So the lemma by Kennedy can be applied again to find the RHS, which give these two unknowns. After repeating these steps an infinite number of times, the R-matrix is fully constructed. Although that sounds a bit unpractical, it may very well happen that after just a few iterations, we can make an educated guess of the matrix elements as functions of the spectral parameter from the first few terms of their Taylor expansions.[2] When this happens, there is also hope to obtain the 2D classical statistical mechanical model dual to the quantum Hamiltonian. This could be a future direction worth exploring for those quantum integrable models that do not currently have a known classical counterpart.

The presentation so far has been assuming a Hamiltonian to be integrable, from which identities like (6) and (9) are derived as consequences. Now suppose instead we are given a Hamiltonian and our goal is to find out whether it is integrable or not. For instance, to check Reshetikhin's condition (6), first construct the candidate two-local operator

$$\tilde{j}_x = \text{tr}_{x+1}[h_{x-1,x} + h_{x,x+1}, \rho_x] + \text{tr}_{x+1,x+2}[h_{x,x+1} + h_{x+1,x+2}, \rho_{x+1}] \tag{10}$$

where tr_x is the partial trace over the local Hilbert space at site x divided by its dimensionality, such that $\text{tr}_x 1_x = 1$. Again, this 'surface flux' $\tilde{j}_x$ (as a function of c) can be calculated just as well for non-integrable Hamiltonians. So the real test is to check if the identity

$$[h_{x-1,x} + h_{x,x+1}, \rho_x] = \tilde{j}_x \otimes 1_{x+1} - 1_{x-1} \otimes \tilde{j}_{x+1} \tag{11}$$

holds. If the LHS indeed turns out to be a pure divergence, since the RHS of (6) involves only known operators except $\check{R}^{(3)}_{x,x+1}$, it can be solved from $\tilde{j}_x$ (as a function of c). Notice that different choices of naturally results in different $\check{R}^{(3)}_{x,x+1}$, which would affect the LHS of (9) at $m = 2$. Suppose for $m = 2$ and $x = 2$, (9) is given by $-D_{123}(c) = X_{12}(c) - X_{23}(c)$, then Kennedy's lemma gives a candidate $\tilde{X}_{12}(c) = -\text{tr}_3 D_{123}(c) - \text{tr}_{34} D_{234}(c)$. The condition at $m = 2$ therefore amounts to $-D_{123}(c) = \tilde{X}_{12}(c) - \tilde{X}_{23}(c)$, which gives a set of equations on the unknown c when considered element-wise. As long as there is a solution for c, the additional condition is considered satisfied up to this step, and one can carry on to higher order conditions fixing c to the solution. On the other hand, if there is no solution, the outcome of the integrability test is conclusively negative.

After understanding the role of the constant c, we can modify the above procedure to find integrable Hamiltonians. We start with a parametrization of generic Hamiltonians subject to certain symmetry, such that each matrix element of $h_{x,x+1}$ is a multivariate function. Then

---

[2]Alternatively, one can trade the infinite elementary steps for one step of solving Sutherland's equation to find the R-matrix [11, 12]. This approach works also for non-relativistic R-matrices, and has been used to classify integrable models with low-dimensional local Hilbert space [13, 14].

the above procedure at each step would result in a set of equations on these parameters. Solving (11) as equations on these unknown variables tells us which particular choices in the parameter space are integrable.

While it has proved to be a very affordable symbolic computation using `Mathematica`, even for large local Hilbert space dimension and to high orders, it seems that the criteria need to be checked to infinite order to be sure that the Hamiltonian is integrable in theory. Surprisingly, in practice all non-integrable Hamiltonians fail the test already at the first order. Put differently, there is no known integrable Hamiltonian that satisfy Reshetikhin's condition, but fails this integrability test at a higher order. Notice that the conservation of the higher order charges to be discussed next does not automatically guarantee the satisfaction of higher order Reshetikhin conditions, as the monodromy matrix $\mathfrak{T}(\xi)$ can be obtained from the R-matrix, but not vice versa.

4 $\mathfrak{sl}(n, \mathbb{C})$ **Hamiltonians**

It is helpful for understanding the previous discussions to parametrize the Hamiltonian in terms of $\mathfrak{sl}(n, \mathbb{C})$ matrices. For a local Hilbert space with n dimensions, any local operator M can be expressed as

$$M = \sum_{\alpha=1}^{n^2-1} m_\alpha T^\alpha + m_0 1_{n \times n}, \tag{12}$$

where the matrices $T^\alpha, \alpha = 1, \cdots n^2-1$ form a basis of traceless $n \times n$ matrices. They can be the Cartan-Weyl or Chevalley basis for example. However, since $\mathfrak{sl}(n, \mathbb{C})$ is the complexification of $\mathfrak{su}(n)$, it is more convenient to let the coefficients m_α be complex numbers, so that the generators T^α become Hermitian matrices forming a representation of $\mathfrak{su}(n)$ and satisfying the commutation and anti-commutation relations

$$[T^\alpha, T^\beta] = if_\gamma^{\alpha\beta} T^\gamma, \quad \{T^\alpha, T^\beta\} = 2\delta^{\alpha\beta} 1_{n \times n} + 2d_\gamma^{\alpha\beta} T^\gamma, \tag{13}$$

with the Kronecker delta $\delta^{\alpha\beta} = 1$ if $\alpha = \beta$; and 0 if $\alpha \neq \beta$. $d_\gamma^{\alpha\beta}$ is totally symmetric about all three indices, while the structure constants $f_\gamma^{\alpha\beta}$ is totally antisymmetric, and satisfy the Jacobi identity[3]

$$f_\delta^{\alpha\beta} f_\epsilon^{\delta\gamma} + f_\delta^{\beta\gamma} f_\epsilon^{\delta\alpha} + f_\delta^{\gamma\alpha} f_\epsilon^{\delta\beta} = 0. \tag{16}$$

Thus any bi-local Hamiltonian operator can be written as

$$h_{j,j+1} = \tilde{a}_{\alpha\beta} \tilde{T}_j^\alpha \tilde{T}_{j+1}^\beta + \tilde{b}_\alpha (\tilde{T}_j^\alpha + \tilde{T}_{j+1}^\alpha) + c, \tag{17}$$

where from now on summation over dummy Greek indices are implied. Parity symmetry requires also that $\tilde{a}_{\beta\alpha} = \tilde{a}_{\alpha\beta}$.[4] This means we can change to the basis in which the matrix

---

[3]Furthermore, the graded Jacobi identities

$$[\{T^\alpha, T^\beta\}, T^\gamma] + [\{T^\beta, T^\gamma\}, T^\alpha] + [\{T^\gamma, T^\alpha\}, T^\beta] = 0,$$
$$[\{T^\alpha, T^\beta\}, T^\gamma] + \{[T^\gamma, T^\beta], T^\alpha\} + \{[T^\gamma, T^\alpha], T^\beta\} = 0 \tag{14}$$

gives the properties (which has not been used though in the following derivations)

$$d_\delta^{\alpha\beta} f_\epsilon^{\delta\gamma} + d_\delta^{\beta\gamma} f_\epsilon^{\delta\alpha} + d_\delta^{\gamma\alpha} f_\epsilon^{\delta\beta} = 0$$
$$d_\delta^{\alpha\beta} f_\epsilon^{\delta\gamma} + f_\delta^{\gamma\beta} d_\epsilon^{\delta\alpha} + f_\delta^{\gamma\alpha} d_\epsilon^{\delta\beta} = 0. \tag{15}$$

[4]Notice that this puts no constraint on the Hermiticity of the Hamiltonian $h_{j,j+1}$, since T^α are chosen to be Hermitian themselves. Non-Hermitian Hamiltonians instead have $a_{\alpha\beta}^* \neq a_{\alpha\beta}$ or $b_\alpha^* \neq b_\alpha$, which is not excluded from the following discussions.

of coefficients become diagonal. Since linear superpositions of traceless Hermitian matrices remain traceless and Hermitian, the structure (13) still holds, with modified constants. Hence in the diagonal basis, the Hamiltonian (17) can be written as

$$h_{j,j+1} = a_\alpha T_j^\alpha T_{j+1}^\alpha + b_\alpha(T_j^\alpha + T_{j+1}^\alpha) + c. \tag{18}$$

The commutator between neighboring Hamiltonians can be written as

$$[h_{12}, h_{23}] = i a_\alpha a_\gamma f_\beta^{\alpha\gamma} T_1^\alpha T_2^\beta T_3^\gamma + i a_\alpha b_\gamma f_\beta^{\alpha\gamma} T_1^\alpha T_2^\beta + i a_\beta b_\gamma f_\alpha^{\gamma\beta} T_2^\alpha T_3^\beta. \tag{19}$$

To get the next commutators, we need

$$
\begin{aligned}
[T_j^\alpha \otimes T_{j+1}^\beta, T_j^\gamma \otimes T_{j+1}^\delta] &= \frac{1}{2}\left([T_j^\alpha, T_j^\gamma] \otimes \{T_{j+1}^\beta, T_{j+1}^\delta\} + \{T_j^\alpha, T_j^\gamma\} \otimes [T_{j+1}^\beta, T_{j+1}^\delta]\right)\\
&= i\left(f_\epsilon^{\alpha\gamma} d_\zeta^{\beta\delta} + f_\zeta^{\beta\delta} d_\epsilon^{\alpha\gamma}\right) T_j^\epsilon T_{j+1}^\zeta + i f_\epsilon^{\alpha\gamma} \delta^{\beta\delta} T_j^\epsilon + i f_\epsilon^{\beta\delta} \delta^{\alpha\gamma} T_{j+1}^\epsilon.
\end{aligned}
\tag{20}
$$

Tedious but elementary derivations show that terms on the LHS of (6) can be put in the form

$$
\begin{aligned}
[h_{12},[h_{12},h_{23}]] &= u_{\alpha\beta\gamma} T_1^\alpha T_2^\beta T_3^\gamma + v_{\alpha\beta} T_1^\alpha T_2^\beta + w_{\alpha\beta} T_1^\alpha T_3^\beta + x_{\alpha\beta} T_2^\alpha T_3^\beta + y_\alpha T_1^\alpha + z_\alpha T_2^\alpha,\\
[h_{23},[h_{23},h_{12}]] &= u_{\gamma\beta\alpha} T_1^\alpha T_2^\beta T_3^\gamma + x_{\beta\alpha} T_1^\alpha T_2^\beta + w_{\alpha\beta} T_1^\alpha T_3^\beta + v_{\beta\alpha} T_2^\alpha T_3^\beta + z_\alpha T_2^\alpha + y_\alpha T_3^\alpha,
\end{aligned}
\tag{21}
$$

with the coefficients after simplifications using the symmetry properties of the indices and (16) becoming

$$
\begin{aligned}
u_{\alpha\beta\gamma} &= a_\gamma a_\delta a_\epsilon f_\zeta^{\gamma\delta}(f_\alpha^{\epsilon\delta} d_\beta^{\epsilon\zeta} + f_\beta^{\epsilon\zeta} d_\alpha^{\epsilon\delta}) + a_\gamma b_\delta(a_\alpha f_\gamma^{\delta\epsilon} f_\beta^{\alpha\epsilon} + a_\epsilon f_\beta^{\gamma\epsilon} f_\alpha^{\delta\epsilon}),\\
v_{\alpha\beta} &= a_\gamma a_\delta b_\epsilon f_\zeta^{\epsilon\delta}(f_\alpha^{\gamma\delta} d_\beta^{\gamma\zeta} + f_\beta^{\gamma\zeta} d_\alpha^{\gamma\delta}) + b_\gamma b_\delta(a_\epsilon f_\beta^{\gamma\epsilon} f_\alpha^{\delta\epsilon} - a_\alpha f_\alpha^{\gamma\epsilon} f_\beta^{\delta\epsilon}),\\
w_{\alpha\beta} &= a_\beta a_\gamma a_\delta f_\alpha^{\gamma\delta} f_\beta^{\delta\gamma},\\
x_{\alpha\beta} &= a_\beta b_\gamma b_\delta f_\alpha^{\gamma\epsilon} f_\beta^{\delta\epsilon},\\
y_\alpha &= a_\beta a_\gamma b_\delta f_\alpha^{\beta\gamma} f_\delta^{\gamma\beta},\\
z_\alpha &= a_\beta a_\beta b_\gamma f_\alpha^{\beta\delta} f_\gamma^{\beta\delta}.
\end{aligned}
\tag{22}
$$

Now using Kennedy's trace trick and the traceless property of T_j^α, we have

$$
\begin{aligned}
X_{12} &= -\operatorname{tr}_3 D_{123} - \operatorname{tr}_{34} D_{234} = (x_{\beta\alpha} - v_{\alpha\beta}) T_1^\alpha T_2^\beta - y_\alpha(T_1^\alpha + T_2^\alpha),\\
X_{23} &= \operatorname{tr}_1 D_{123} + \operatorname{tr}_{01} D_{012} = (x_{\alpha\beta} - v_{\beta\alpha}) T_2^\alpha T_3^\beta - y_\alpha(T_2^\alpha + T_3^\alpha)
\end{aligned}
\tag{23}
$$

In order for them to be the same operator acting on different degrees of freedom, we must have $v_{\alpha\beta} + x_{\alpha\beta} = v_{\beta\alpha} + x_{\beta\alpha}$, $\forall \alpha, \beta$. In addition, $-D_{123} = X_{12} - X_{23}$ requires $u_{\alpha\beta\gamma} = u_{\gamma\beta\alpha}$, $\forall \alpha, \beta, \gamma$. Explicitly, they are written as

$$
\left(f_\alpha^{\epsilon\delta} d_\beta^{\epsilon\gamma} - f_\beta^{\epsilon\delta} d_\alpha^{\epsilon\gamma}\right) f_\zeta^{\gamma\delta}\left(a_\gamma + a_\delta\right) a_\epsilon b_\zeta = 0, \quad \forall \alpha \neq \beta,
$$

$$
\begin{aligned}
a_\delta a_\epsilon \Big[d_\beta^{\delta\zeta}\left(a_\gamma f_\zeta^{\gamma\epsilon} f_\delta^{\alpha\epsilon} - a_\alpha f_\zeta^{\alpha\epsilon} f_\delta^{\gamma\epsilon}\right) + f_\beta^{\delta\zeta}\left(a_\gamma f_\zeta^{\epsilon\gamma} d_\alpha^{\delta\epsilon} - a_\alpha f_\zeta^{\epsilon\alpha} d_\gamma^{\delta\epsilon}\right)\Big]&\\
+ a_\epsilon b_\delta\left(a_\gamma f_\beta^{\epsilon\gamma} f_\alpha^{\delta\epsilon} - a_\alpha f_\beta^{\alpha\epsilon} f_\gamma^{\epsilon\delta}\right) = 0, \quad \forall \beta, \text{ and } \alpha \neq \gamma.
\end{aligned}
\tag{24}
$$

To understand these conditions, let us look at an example of $n = 2$. In this case, $d_\gamma^{\alpha\beta} = 0$, so the only condition left is $a_\epsilon b_\delta\left(a_\gamma f_\beta^{\epsilon\gamma} f_\alpha^{\delta\epsilon} - a_\alpha f_\beta^{\alpha\epsilon} f_\gamma^{\epsilon\delta}\right) = 0$. Due to the total antisymmetry

of the structure constant, and between the indices α and γ, there are six conditions that have to be satisfied simultaneously

$$a_1 a_2 b_1 = a_1 a_2 b_2 = a_1 a_3 b_1 = a_1 a_3 b_3 = a_2 a_3 b_2 = a_2 a_3 b_3 = 0. \qquad (25)$$

The solutions can be classified as follows: In the case $b_1 = b_2 = b_3 = 0$, there is no constraint on $a_{1,2,3}$, meaning that XYZ models are integrable. In the case $b_1 = b_2 = 0$, we need further either $a_1 = a_2 = 0$, giving the longitudinal field Ising model, or $a_3 = 0$, known as the XYh model. The case $b_3 = 0$ has to be accompanied by $a_1 = a_2 = 0$, which is the transverse field Ising model. Finally, if all three linear terms are present, the Hamiltonian can only be integrable if it is non-interacting, that is, $a_1 = a_2 = a_3 = 0$. These are all of the integrable models with two local degrees of freedom, as have been well classified using the conserved charge approach [15, 16].

The full classification of integrable models with larger local Hilbert space is left for future works, but a simple observations can be made. If there is no linear terms in the Hamiltonian, the condition becomes

$$a_\delta a_\epsilon \left[d_\beta^{\delta\zeta} \left(a_\gamma f_\zeta^{\epsilon\gamma} f_\alpha^{\delta\epsilon} - a_\alpha f_\zeta^{\alpha\epsilon} f_\gamma^{\epsilon\delta} \right) + f_\beta^{\delta\zeta} \left(a_\gamma f_\zeta^{\epsilon\gamma} d_\alpha^{\delta\epsilon} - a_\alpha f_\zeta^{\epsilon\alpha} d_\gamma^{\delta\epsilon} \right) \right] = 0, \qquad (26)$$

which is a set of equations on the coefficients a_α. So anisotropic higher spin chains and Hamiltonians explicitly breaking SU(N) symmetry are generically non-integrable, as has been studied in Ref. [3, 8].

In Appendix B, the construction in this section will be used to compute the explicit expression for $\check{R}_{12}^{(3)} = X_{12} + h_{12}^3$ for a few examples of integrable models.

5 Conserved charges and the boost operator

In recent decades, quantum integrability has been much more heavily investigated in terms of symmetry currents and conserved charges than in the framework of YBE [17]. So this section is dedicated to making a connection between the two contexts highlighting the fact that the YBE is a stronger condition than existence of infinite conserved charges.

From the R-matrix obtained in the previous section, one can construct the monodromy matrix $\mathfrak{T}(\xi) = \prod_x R_{x,x+1}(\xi),$[5] where $R_{x,x+1}(\xi) = P_{x,x+1} \check{R}_{x,x+1}(\xi)$, with $P_{x,x+1}$ being the permutation between site x and $x+1$. One can check that the YBE in the form

$$R_{x,x+1}(\zeta) R_{x-1,x+1}(\xi) R_{x-1,x}(\xi - \zeta) = R_{x-1,x}(\xi - \zeta) R_{x-1,x+1}(\xi) R_{x,x+1}(\zeta). \qquad (27)$$

ensures the commutativity between monodromy operators with different spectral parameters $[\mathfrak{T}(\xi), \mathfrak{T}(\zeta)] = 0$. Therefore, the charges defined by

$$\ln \mathfrak{T}(\xi) = \sum_{n=0}^{\infty} \frac{\xi^n}{n!} Q^{(n+1)}, \text{ or } Q^{(n+1)} = \frac{d^n}{d\xi^n} \ln \mathfrak{T}(\xi) \Big|_{\xi=0}$$

all commute $[Q^{(m)}, Q^{(n)}] = 0$, with the first three being $Q^{(1)} = P \equiv \sum_x \ln P_{x,x+1}$, $Q^{(2)} = H$ and $Q^{(3)} = Q$. This way the explicit forms of the higher order conserved charges can always be obtained from the expansion of the vertex operator when the YBE is satisfied, even though their densities are not readily available.[6]

---

[5]For a finite chain with periodic boundary condition, the trace can to be taken to get the column transfer matrix.

[6]This is because the support of the densities do not grow as fast as one would expect from order to order due to similar cancellations as in (8).
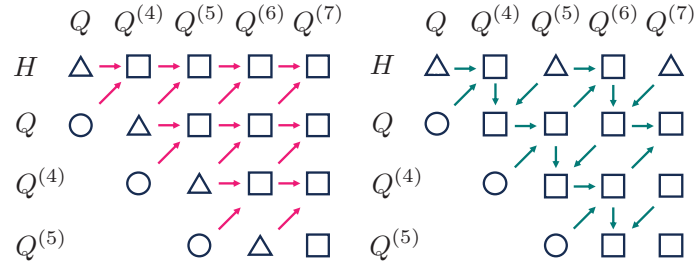
Figure 1: Two alternative schematics of the mutual commutativity of conserved charges: Circles represent the commutation between a charge with itself, which are trivially satisfied; triangles represent the independent additional conditions that imply all the other commutativity represented by squares.

A more convenient way to obtain the explicit form of charges is via the boost operator [18], defined as $B = \sum_x x h_{x-1,x}$. The name is fitting for its operation on the monodromy matrix, if its spectral parameter is interpreted as a rapidity $[B, \mathfrak{T}(\xi)] = \partial_\xi \mathfrak{T}(\xi)$. It follows that the boost operator acts as a ladder operator on the infinite sequence of conserved charges

$$[B, Q^{(n)}] = Q^{(n+1)}, \quad n = 1, 2, 3, \cdots. \tag{28}$$

To reveal the analogy with the generator of the Lorentz boost $K(t) = \int dx \big(x H(x) - t P(x) \big)$ in field theory, one can look at the boost operator in the Heisenberg picture, which is referred to as a time-dependent symmetry generator in the mathematics literature [19]

$$B(t) = e^{itH} B e^{-itH} = \sum_{n=0}^\infty \frac{(it)^n}{n!} \operatorname{ad}_H^n B \tag{29}$$

evaluated at $t = 0$. If the first non-trivial charge $Q = [B, H]$ generated by B is already conserved, i.e. $[H, [H, B]] = 0$, the infinite sum terminates, and the above definition becomes

$$B(t) = B - it Q = \sum_x \big(x h_{x-1,x} - it \rho_x \big). \tag{30}$$

Notice that the interpretation of the analogy to Lorentz boost is different from that given in Ref. [20], which focused instead on the boost between momentum and energy. By definition of the boost operator, the relation $[B, P] = H$ always holds, regardless of the integrability of the Hamiltonian. But as shown here, $B(t)$ only becomes a time-dependent symmetry if the infinite formal sum (29) is meaningful, making it a master symmetry, when the three-local charge generated by B is conserved. The interpretation of the boost operator serves as a first hint that there is a discrete Lorentz symmetry whenever Reshetikhin's condition holds, and the existence of all the higher conserved charges are automatically ensured by this symmetry. In Sec. 6, we will take a closer look at how such Lorentz invariance works out when combined with lattice translation into a discrete Poincaré group.

Unlike the definitions of classical integrability, which accommodate a spectrum of different degrees of integrability, such as integrable by quadrature, requiring the first integrals to form a closed Lie subalgebra instead of involutive, quantum integrability is usually defined by the mutual commutativity of all conserved charges, not the least due to the closely-knit YBE and transfer matrix formalism. Because of the ladder property of the boost operator, not all these commutativity are independent. This can be seen by applying the Jacobi identity

$$[Q^{(m+1)}, Q^{(n)}] = [Q^{(n+1)}, Q^{(m)}] + [B, [Q^{(m)}, Q^{(n)}]]. \tag{31}$$

Hence, at most an $\mathcal{O}(N)$ number of the total $\mathcal{O}(N^2)$ commutativity can be independent for N conserved charges.[7] Two alternative choices of them, either $[Q^{(n)}, Q^{(n+1)}] = 0$ for all $n \geq 2$, or $[H, Q^{(2m+1)}] = 0$ for all $m \geq 1$, are summarized in Fig. 1.

As mentioned earlier, (9) is believed to be in one-to-one correspondence with the independent inputs of the mutual commutativity of all the conserved charges in Fig. 1, although the exact derivation from one to the other remains to be established. Since the conservation of the lowest order charge Q is also equivalent to the Reshetikhin condition, it is no surprise that in all known examples all the higher order commutation relations are automatically satisfied when $[H, Q] = 0$. Hence the Reshetikhin condition has been conjectured by Grabowski and Mathieu as a sufficient condition for integrability [2]. In a highly technical proof, the conjecture seems to have been established recently [23]. However, showing that the Reshetikhin condition implies (9) and hence the YBE remains an open question, as the derivation of the conserved charges from the YBE seems like a one-way road.

6   The lattice Poincaré group

Thacker argued that the algebraic structure of the conserved charges (28) is the infinite dimensional lattice analog of the Poincaré algebra [20]

$$[P, H] = 0, \quad [K, P] = iH, \quad [K, H] = iP, \tag{32}$$

where in the continuous limit the odd (resp. even) order charges converge to P (resp. H). As appealing as it is, the analogy is a bit vague: While the Poincaré symmetry is a kinetic or spacetime symmetry, the conserved charges of integrable spin chains generate a dynamic symmetry in the internal degree of freedoms in the Hilbert space. More direct analogies were later studied as κ-deformed Poincaré algebras [24, 25], where the deformation parameter is related to the lattice spacing, and instead of an infinite set of generators, the group is generated by the enveloping algebra of the three operators that generate the continuous group (32) [26].

Following Ref. [27], I demonstrate now how the infinite algebraic structure of (28) naturally arises when (1+1)D Lorentz invariance is combined with discrete translation invariance. Since at least the translation in one direction is already a discrete subgroup, it is better to work with group elements instead of the Lie algebra that generates a continuous group. As a semi-direct product, the Poincaré group has the multiplication rule

$$\big(\Lambda(\eta), \vec{\alpha}\big) \cdot \big(\Lambda(\theta), \vec{\beta}\big) = \big(\Lambda(\eta + \theta), \vec{\alpha} + \Lambda(\eta)\vec{\beta}\big), \tag{33}$$

where the Lorentz boost has the two-dimensional representation

$$\Lambda(\eta) = \begin{pmatrix} \cosh\eta & \sinh\eta \\ \sinh\eta & \cosh\eta \end{pmatrix}. \tag{34}$$

The (1+1)D lattice Poincaré group is generated by a finite translation $t_1 = \big(\Lambda(0), (1\ 0)^T\big)$ and a finite boost $b(\eta) = \big(\Lambda(\eta), (0\ 0)^T\big)$. The group commutator between the two produces a translation in another direction

$$t_2 = [b(\eta), t_1] = b(\eta) t_1 b(\eta)^{-1} t_1^{-1} = \Big(\Lambda(0), \begin{pmatrix} \cosh\eta - 1 \\ \sinh\eta \end{pmatrix}\Big). \tag{35}$$

We are free to choose the new direction as the second basis vector that together span the (1+1)D spacetime. But the next commutator

$$t_3 = [b(\eta), t_2] = b(\eta) t_2 b(\eta)^{-1} t_2^{-1} = (t_1 t_2)^{2(\cosh\eta - 1)} \tag{36}$$

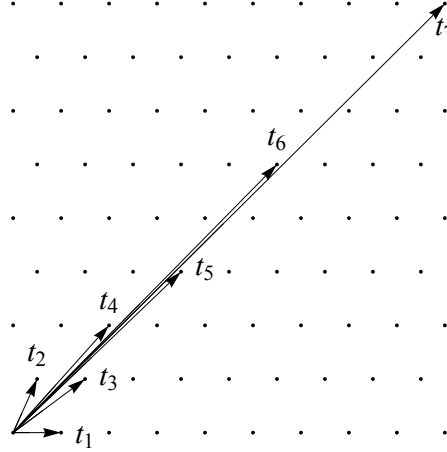[7]This has been noticed before in Ref. [21, 22], among many others.

Figure 2: The lattice vectors generated by boosting the translation t_1 with the minimal rapidity $\operatorname{arcosh} \frac{3}{2}$. They alternate between space-like and time-like vectors and approach the light-cone in the infinite limit.

has to end up on a lattice point, as required by the group closure. This only happens if $\cosh \eta$ take positive half-integer values. $\cosh \eta = 1$ is already in the group as the identity element. Among the rest of the possibilities, we can only pick one, as the product of two different boosts would end up outside the lattice. The choice of η labels the irreducible representation.

Now that the action of t_3 remains on the lattice, it follows by induction that all higher group commutators $t_{k+1} = [b(\eta), t_k]$ land on the lattice as well. In particular, for $\cosh \eta = \frac{3}{2}$, we have $t_{k+1} = t_k t_{k-1}$, and the corresponding lattice vectors form a Fibonacci sequence, as depicted in Fig. 2. By analogy, the conservation of the higher conserved charges encountered in Sec. 5 could be understood as the consequence of a discrete Poincaré symmetry, which depends on the conservation of the three-local charge alone.

7 The discrete conformal algebra

In order to move from the lattice Poincaré group of spacetime coordinates to a discrete Poincaré algebra for conserved charges in quantum integrability, we need to find the generator of discrete translation, in a way that is compatible with the algebra among infinitesimal generators of a Lie group.[8] This can be done by introducing the superoperator

$$\mathcal{P}[j_x^{(n)}] = (1 - \mathcal{T})[j_x^{(n)}] = j_x^{(n)} - j_{x+1}^{(n)}, \tag{37}$$

where $\mathcal{T}[j_x] = \operatorname{Ad}_T j_x = T j_x T^{-1}$ shifts the support of an operator by one lattice spacing. A formal inverse of the difference operator can be introduced to solve the symmetry currents from the continuity equation

$$\frac{d\rho_x^{(n)}}{dt} = i[H, \rho_x^{(n)}] = j_x^{(n)} - j_{x+1}^{(n)} \tag{38}$$

satisfied by the higher charge densities. Say the n-local density operator $\rho_x^{(n)}$ is supported by lattice sites from $x - n + 2$ to $x + 1$, and the $(n+1)$-local current operator $j_x^{(n)}$ acts non-trivially

---

[8]This has been done in the context of single-body quantum mechanical integrable systems [28], and the proper mathematical tool is the umbral calculus [29–33].

on sites $x - n + 1, \cdots, x$, then by a generalized version of Kennedy's inversion formula (10),

$$
\begin{aligned}
j_x^{(n)} &= \sum_{r=0}^{\infty} \text{tr}_{x+1,\cdots,x+n-2+r} \frac{d\rho_{x+r}^{(n)}}{dt} \\
&= \sum_{r=0}^{\infty} (\text{tr}_{x+1}\,\mathcal{T})^r \, \text{tr}_{x+1} \frac{d\rho_x^{(n)}}{dt} \\
&= (1 - \text{tr}_{x+1}\,\mathcal{T})^{-1} \, \text{tr}_{x+1} \frac{d\rho_x^{(n)}}{dt} \equiv \mathcal{P}^{-1}\Big[\frac{d\rho_x^{(n)}}{dt}\Big].
\end{aligned}
\tag{39}
$$

We do not need to worry about the convergence of the infinite sum because it terminates due to the cyclic property of trace after it is taken over the entire support of $[H, \rho_x^{(n)}]$.

One can also introduce superoperators that correspond to a scaling transformation

$$
\mathcal{D} = x(\mathcal{T}^{-1} - 1),
\tag{40}
$$

and a special conformal transformation [34]

$$
\mathcal{K} = x(x-1)\mathcal{T}^{-1}(\mathcal{T}^{-1} - 1).
\tag{41}
$$

Then using the canonical commutation relation of finite difference $[\mathcal{T} - 1, x\mathcal{T}^{-1}] = 1$, they can be shown to obey the commutation rules of an $\mathfrak{sl}_2$ algebra

$$
[\mathcal{D}, \mathcal{P}] = \mathcal{P}, \quad [\mathcal{D}, \mathcal{K}] = -\mathcal{K}, \quad [\mathcal{K}, \mathcal{P}] = 2\mathcal{D}.
\tag{42}
$$

Together with $\mathcal{D}[H] = H$, $\mathcal{K}[H] = 2B$, and $\mathcal{P}[B] = H$, they belong to a (1+1)D lattice conformal algebra.[9]

8   The generalized Reshetikhin condition

One outdated counterexample to conserved charges being generated by the boost operator defined above is the Hubbard model. As noted since the early days, combining a charge and a spin sector, it is not a 'relativistic' integrable model, meaning the R-matrix associated with its Hamiltonian depends on both of the spectral parameters, instead of just the difference of two 'rapidities'. Put differently, the gapless modes in the two sectors can propagate at different speeds, which necessarily breaks the lattice Lorentz invariance that the boost operator derived its name from. Yet, a ladder operator that generalizes the boost operator has been found for such models [35], by expanding the R-matrix in terms of the difference between the two spectral parameters, such that the coefficients become functions of one of them

$$
\check{R}_{x,x+1}(\mu, \nu) = 1_{x,x+1} + \sum_{n=1}^{\infty} \frac{(\mu - \nu)^n}{n!} \check{R}_{x,x+1}^{(n)}(\nu).
\tag{43}
$$

As a result, the Hamiltonian and its constant shift also belong to a class labeled by one of the spectral parameters $\check{R}_{x,x+1}^{(1)}(\nu) = h_{x,x+1}(\nu) + c(\nu)1_{x,x+1}$. The general YBE in braided form becomes

$$
\check{R}_{x,x+1}(\nu, \lambda)\check{R}_{x-1,x}(\nu, \mu)\check{R}_{x,x+1}(\lambda, \mu) = \check{R}_{x-1,x}(\lambda, \mu)\check{R}_{x,x+1}(\nu, \mu)\check{R}_{x-1,x}(\nu, \lambda).
\tag{44}
$$

Taking $\mu = \nu$, it implies the unitarity

$$
\check{R}_{x,x+1}(\mu, \nu)\check{R}_{x,x+1}(\nu, \mu) = 1_{x,x+1}.
\tag{45}
$$

---

[9]In addition, we also have $\mathcal{D}[B] = \frac{1}{2}\mathcal{D}\big[\mathcal{K}[H]\big] = \frac{1}{2}[\mathcal{D}, \mathcal{K}][H] + \frac{1}{2}\mathcal{K}\big[\mathcal{D}[H]\big] = 0$.

Like before, unitarity provides a way to determine even order terms in (43) from lower order ones. In particular, $\check{R}^{(2)}_{x,x+1}(\nu) = \left(\check{R}^{(1)}_{x,x+1}(\nu)\right)^2 + \partial_\nu \check{R}^{(1)}_{x,x+1}(\nu)$.

The coefficient of $\mu \nu^2$ in (44) (after Taylor expanding $\check{R}^{(n)}(\lambda)$ around $\lambda = \mu$) gives the generalized Reshetikhin condition [35]

$$[h_{12} + h_{23}, [h_{23}, h_{12}]] + [h_{12}, h'_{23}] + [h_{23}, h'_{12}] + \frac{1}{2}\left([h_{12}, h'_{12}] + [h_{23}, h'_{23}]\right) = Y_{12} - Y_{23}. \quad (46)$$

where $Y_{12} = \check{R}^{(3)}_{12}(\mu) - h^3_{12}(\mu) - \partial^2_\mu h_{12}(\mu) - \frac{3}{2}\{h_{12}(\mu), \partial_\mu h_{12}(\mu)\}$.[10] This can be viewed as an equation of an unknown matrix that is the first order derivative of the Hamiltonian density.

In order to see how $h'_{12}(0) = \partial_\mu h_{12}(\mu)|_{\mu=0}$ could possibly be solved, we can parametrize again the Hamiltonian in terms of $\mathfrak{su}(n)$ operators in Sec. 4, with a_α, b_α, c being functions of the spectral parameter instead of constants. This means that $\partial_\mu h_{j,j+1}$ is also parametrized as

$$h'_{j,j+1} = a'_\alpha T^\alpha_j \otimes T^\alpha_{j+1} + b'_\alpha \left(T^\alpha_j + T^\alpha_{j+1}\right) + c'(\mu). \quad (47)$$

The additional terms in (46) compared to (6) are given by

$$[h_{12}, h'_{23}] + [h_{23}, h'_{12}]$$
$$= i(a_\alpha a'_\gamma - a'_\alpha a_\gamma) f^{\alpha\gamma}_\beta T^\alpha_1 T^\beta_2 T^\gamma_3 + i(a_\alpha b'_\gamma - a'_\alpha b_\gamma) f^{\alpha\gamma}_\beta T^\alpha_1 T^\beta_2 + i(a'_\beta b_\gamma - a_\beta b'_\gamma) f^{\gamma\beta}_\alpha T^\alpha_2 T^\beta_3$$
$$+ 2i b_\beta b'_\gamma f^{\beta\gamma}_\alpha T^\alpha_2,$$
$$\frac{1}{2}\left([h_{12}, h'_{12}] + [h_{23}, h'_{23}]\right) \quad (48)$$
$$= \frac{i}{2}\left(a_\gamma a'_\delta (f^{\gamma\delta}_\alpha d^{\gamma\delta}_\beta + f^{\gamma\delta}_\beta d^{\gamma\delta}_\alpha) + \left((a'_\alpha - a'_\beta) b_\gamma - (a_\alpha - a_\beta) b'_\gamma\right) f^{\alpha\beta}_\gamma\right)\left(T^\alpha_1 T^\beta_2 + T^\alpha_2 T^\beta_3\right)$$
$$+ \frac{i}{2} b_\beta b'_\gamma f^{\beta\gamma}_\alpha (T^\alpha_1 + 2T^\alpha_2 + T^\alpha_3).$$

So the generalized Reshetikhin condition is equivalent to

$$u_{\gamma\beta\alpha} - u_{\alpha\beta\gamma} + i f^{\alpha\gamma}_\beta (a_\alpha a'_\gamma - a_\gamma a'_\alpha) = 0, \qquad \forall \alpha, \beta, \gamma,$$
$$x_{\beta\alpha} + v_{\beta\alpha} - x_{\alpha\beta} - v_{\alpha\beta} + i a_\gamma a'_\delta (f^{\gamma\delta}_\alpha d^{\gamma\delta}_\beta + f^{\gamma\delta}_\beta d^{\gamma\delta}_\alpha) = 0, \qquad \forall \alpha, \beta, \quad (49)$$
$$b_\beta b'_\gamma f^{\beta\gamma}_\alpha = 0, \qquad \forall \alpha.$$

Compared to (24), which contains no unknown variables, (49) can be treated as equations on a'_α and b'_α. It is not guaranteed that a solution exists. But when it does, it could be a sign of integrability. Of course, generalized version of the higher order conditions (9) still needs to be checked to conclusively claim integrability.[11]

Since the section started with a general form of the YBE, without the relativistic assumption, the generalized Reshetikhin condition could even be conjectured as a necessary condition for quantum integrability, at least in the YBE sense. However, to bootstrap the full parameter dependent integrable class of Hamiltonians $H(\mu)$ therefrom seems unlikely, not to mention the R-matrix, as all that we could solve in the above procedure is $\partial_\mu h(\mu)|_{\mu=0}$ and $\check{R}^{(3)}_{12}(0) - \partial^2_\mu h_{12}(\mu)|_{\mu=0}$.

---

[10]The apparent difference from Eq. (12) in Ref. [35] is the result of extra minus signs in the odd order terms of the definition (43).

[11]Two comments are in order. First, this serves as a way to determine $h'(0)$ without knowledge of the R-matrix. This was pointed out in a comment following Eq. (16) in Ref. [35] as impossible without knowledge of the whole class of Hamiltonians to which the Hubbard model belongs. The more accurate statement in light of Kennedy's method would be that $h'(\mu)$ cannot be determined without knowing the R-matrix. Second, the above statement is in no contradiction to Theorem 13 of Ref. [23], which assumes that the R-matrix of an integrable Hamiltonian is known. For non-relativistic integrable Hamiltonians, this happens extremely rarely, and has always been considered a great achievement when it does [36,37]. In that case, a full class of Hamiltonians parametrized by the spectral parameter is already given. However, as an integrability test, the starting point here is a generic Hamiltonian isolated in the parameter space.

9 Conclusion

The main result of the article is a bootstrap program for finding the *R*-matrix of a relativistic integrable Hamiltonians. In particular, explicit forms of the higher order Reshetikhin conditions (9) has been obtained. Regardless of the order, they all depend on three-local operators, and should be simpler observables to study than the conserved currents of a generalized Gibbs ensemble [38, 39]. How Reshetikhin's condition is supposed to imply (9) and hence the YBE in practice remains an unanswered question. Future pursuit in this direction would not only further shed light upon how the three-local charge conservation implies existence of all higher order charges, but also ultimately address the question of whether quantum integrability happens always in the YBE sense. For instance, the recent solution of a quasi-1D lattice model was based on the Bethe ansatz, but does not depend on the YBE of scattering matrices [40].

Instead of being considered as conserved charges that protect the integrability of the Hamiltonian, the charges should be treated on the same footing as collectively a class of integrable Hamiltonians that are mutually conserved charges of one another. For the non-relativistic case, each of Hamiltonians in this countable class labeled by an integer that is the order of the charge, is in turn a continuous class of integrable Hamiltonians labeled by one of the spectral parameters of their *R*-matrix.

One possible direction for extension is if instead of $[H, [H, B]] = 0$, one has $\mathrm{ad}_H^n B \neq 0$ and $\mathrm{ad}_H^{n+1} B = 0$, for $n \geq 2$. In that case, $B(t)$ would still be a time-dependent symmetry potentially good for generating a hierarchy of conserved charges, except its explicit form would contain operators with support up to $n+2$. The first conserved charge would instead become $\mathrm{ad}_H^n B$. In the language of mastersymmetry, B would be an H-mastersymmetry of degree n [19]. Perhaps it makes sense to find an inverse of the boost operator that generate lower charges from higher ones, which is also consistent with the viewpoint that all the conserved charges are actually the same charge observed in different reference frames.

It should be noted that despite the lack of a counterexample of non-integrable spin chain that satisfies the Reshetikhin condition but not its higher order cousins, it is known in the mathematics literature that one conserved charge does not imply infinite many for non-linear differential equations [41]. This contrast might hint that the quantum integrable spin chains we are familiar with today might be just a tiny class of more varieties of quantum notions of integrability, as integrable evolution equations with two spatial dimensions is a well studied subject.

Acknowledgements

A Proof of canceling surface fluxes

For simplicity of presentation, in this appendix the shorthand notation $a_n = \check{R}_{12}^{(n)}, b_n = \check{R}_{23}^{(n)}$ is introduced. Expanding the YBE, the coefficient of the $\xi \zeta^{2m}$ term gives

$$\sum_{k=0}^{2m}(-1)^k\binom{2m}{k}(a_k b_1 a_{2m-k} - b_{2m-k}a_1 b_k) = \sum_{k=0}^{2m}(-1)^k\binom{2m}{k}(b_{2m-k}b_{k+1} - a_{k+1}a_{2m-k}). \quad \text{(A.1)}$$

The RHS is in fact the difference of two identical operators acting on two different pairs of neighboring sites:

$$\sum_{k=0}^{2m}(-1)^k\binom{2m}{k}b_{2m-k}b_{k+1}$$

$$=\sum_{k=0}^{2m}(-1)^k\binom{2m}{k}b_k b_{2m-k+1}$$

$$=\sum_{k=1}^{2m}(-1)^k\left(\binom{2m+1}{k}-\binom{2m}{k-1}\right)b_k b_{2m-k+1}+b_{2m+1}$$

$$=\sum_{k=1}^{2m}(-1)^k\binom{2m+1}{k}b_k b_{2m-k+1}-\sum_{k=1}^{2m}(-1)^k\binom{2m}{k-1}b_k b_{2m-k+1}+b_{2m+1}$$

$$=\sum_{k=1}^{2m}(-1)^k\binom{2m+1}{k}b_k b_{2m-k+1}+\sum_{k=0}^{2m-1}(-1)^k\binom{2m}{k}b_{k+1}b_{2m-k}+b_{2m+1}$$

$$=\sum_{k=0}^{2m}(-1)^k\binom{2m}{k}b_{k+1}b_{2m-k},$$

where in the last step Eq. (5) has been used. The LHS of Eq. (A.1) can also be massaged into the form of Eq. (9) by expanding the terms in the summand containing the largest subscript $2m$ using Eq. (4):

$$\sum_{k=0}^{2m}(-1)^k\binom{2m}{k}(a_k b_1 a_{2m-k}-b_{2m-k}a_1 b_k)$$

$$=\sum_{k=1}^{2m-1}(-1)^k\binom{2m}{k}(a_k b_1 a_{2m-k}-b_{2m-k}a_1 b_k)+a_0 b_1 a_{2m}+a_{2m}b_1 a_0-b_{2m}a_1 b_0-b_0 a_1 b_{2m}$$

$$=\sum_{k=1}^{2m-1}(-1)^k\binom{2m}{k}(a_k[b_1,a_{2m-k}]+a_k a_{2m-k}b_1-[b_{2m-k},a_1]b_k-a_1 b_{2m-k}b_k)$$
$$+b_1 a_{2m}+a_{2m}b_1-b_{2m}a_1-a_1 b_{2m}$$

$$=\sum_{k=1}^{2m-1}(-1)^k\binom{2m}{k}(a_k[b_1,a_{2m-k}]-[b_{2m-k},a_1]b_k)+b_1 a_{2m}-a_{2m}b_1-b_{2m}a_1+a_1 b_{2m}$$

$$=\sum_{k=1}^{2m-1}(-1)^k\binom{2m}{k}([a_k,[b_1,a_{2m-k}]]+[b_1,a_{2m-k}]a_k-[[b_{2m-k},a_1],b_k]-b_k[b_{2m-k},a_1])$$
$$+b_1 a_{2m}-a_{2m}b_1-b_{2m}a_1+a_1 b_{2m}$$

$$=\sum_{k=1}^{2m-1}(-1)^k\binom{2m}{k}([a_k,[b_1,a_{2m-k}]]-[b_k,[a_1,b_{2m-k}]])-b_1 a_{2m}-a_{2m}b_1-b_{2m}a_1-a_1 b_{2m}$$

$$-\sum_{k=1}^{2m-1}(-1)^k\binom{2m}{k}(a_{2m-k}b_1 a_k-b_k a_1 b_{2m-k})$$

$$=\sum_{k=1}^{2m-1}(-1)^k\binom{2m}{k}([a_k,[b_1,a_{2m-k}]]-[b_k,[a_1,b_{2m-k}]])$$

$$-\sum_{k=0}^{2m}(-1)^k\binom{2m}{k}(a_k b_1 a_{2m-k}-b_{2m-k}a_1 b_k).$$

From this we get

$$
\sum_{k=0}^{2m} (-1)^k \binom{2m}{k} (a_k b_1 a_{2m-k} - b_{2m-k} a_1 b_k)
$$
$$
= \frac{1}{2} \sum_{k=1}^{2m-1} (-1)^k \binom{2m}{k} ([a_k, [b_1, a_{2m-k}]] - [b_k, [a_1, b_{2m-k}]]).
$$

(A.2)

B Explicit examples of the first steps in the bootstrap program

B.1 Spin-$\frac{1}{2}$ chains

The Heisenberg Hamiltonian is given by taking $T_j^\alpha = \sigma_j^\alpha$, $a_\alpha = 1$, and $b_\alpha = c = 0$ in (18), where the Pauli matrices with structure constants proportional to the Levi-Civita symbol $f_\gamma^{\alpha\beta} = 2\epsilon^{\alpha\beta\gamma}$, and $d_\gamma^{\alpha\beta} = 0$. In this case, (20) is specified to the form

$$
[\sigma_j^\alpha \otimes \sigma_{j+1}^\beta, \sigma_j^\gamma \otimes \sigma_{j+1}^\delta] = 2i\left(\epsilon^{\alpha\gamma\epsilon}\delta^{\beta\delta}\sigma_j^\epsilon + \epsilon^{\beta\delta\epsilon}\delta^{\alpha\gamma}\sigma_{j+1}^\epsilon\right).
$$

Hence for the Heisenberg Hamiltonian,

$$
\begin{aligned}
[h_{12}, h_{23}] &= \sigma_1^\alpha \otimes [\sigma_2^\alpha, \sigma_2^\beta] \otimes \sigma_3^\beta = -2i\epsilon^{\alpha\beta\gamma}\sigma_1^\alpha\sigma_2^\beta\sigma_3^\gamma, \\
[h_{12}, [h_{12}, h_{23}]] &= -2i\epsilon^{\alpha\beta\gamma}[\sigma_1^\delta \otimes \sigma_2^\delta, \sigma_1^\alpha \otimes \sigma_2^\beta] \otimes \sigma_3^\gamma = 4\epsilon^{\alpha\beta\gamma}\left(\epsilon^{\beta\alpha\epsilon}\sigma_1^\epsilon + \epsilon^{\alpha\beta\epsilon}\sigma_2^\epsilon\right)\sigma_3^\gamma \\
&= 8\left(\sigma_2^\alpha\sigma_3^\alpha - \sigma_1^\alpha\sigma_3^\alpha\right), \\
h_{12}^2 &= \sigma_1^\alpha\sigma_1^\beta \otimes \sigma_2^\alpha\sigma_2^\beta = (\delta^{\alpha\beta} + i\epsilon^{\alpha\beta\gamma}\sigma_1^\gamma) \otimes (\delta^{\alpha\beta} + i\epsilon^{\alpha\beta\gamma}\sigma_2^\gamma) = 3 - 2\sigma_1^\alpha\sigma_2^\alpha \\
&= 3 - 2h_{12}, \\
h_{12}^3 &= 3h_{12} - 2h_{12}^2 = 7h_{12} - 6.
\end{aligned}
$$

Kennedy's trace trick then tells us that $X_{12} = 8h_{12}$, giving

$$
R_{12}^{(3)} = X_{12} + (h_{12} + c)^3 = 3(c^2 - 2c + 5)h_{12} + c^3 + 9c - 6.
$$

(B.1)

In this case, the constant c is not going to matter, so we choose $c = 0$ to simplify the following calculations. Eq. (4) further tells us that $\check{R}_{12}^{(4)} = 4h_{12}R_{12}^{(3)} - 3h_{12}^4 = 117 - 84h_{12}$. In this example, it is easy to see that all higher order conditions are automatically guaranteed by the Reshetikhin condition. Indeed, since all higher order coefficients of $\check{R}_{j,j+1}^{(k)}$ and powers of the local Hamiltonian are proportional to the Hamiltonian density itself up to a constant shift. The LHS of (9) are always proportional to the LHS of Reshetikhin condition. At $m = 2$, (9) becomes

$$
384(h_{23} - h_{12}) = 621(h_{12} - h_{23}) - \check{R}_{12}^{(5)} + \check{R}_{23}^{(5)},
$$

(B.2)

giving $\check{R}_{12}^{(5)} = 237h_{12} - 540$. We will not continue further, but simply point out that this is leading to the familiar result that the R-matrix for the Heisenberg Hamiltonian is proportional to the identity operator, making $\check{R}$, in the braided form, proportional to the permutation operator or the Hamiltonian itself up to a constant shift.

Another curious observation that can be made here is

$$
\mathrm{ad}_{h_{12}}^3 h_{23} = -\mathrm{ad}_{h_{23}}^3 h_{12} = 16[h_{12}, h_{23}],
$$

(B.3)

known as the Dolan-Grady (DG) relation [9] satisfied by the Onsager algebra [10].

Finally, symbolic calculation using `Mathematica` gives the following relation for the XYZ Hamiltonian

$$[h_{12} + h_{23}, [h_{12}, h_{23}]] = 2(h_{12}^3 - h_{23}^3) - 2J^2(h_{12} - h_{23}), \tag{B.4}$$

which tells us that for the XYZ Hamiltonian $\check{R}_{12}^{(3)} = 2(J_x^2 + J_y^2 + J_z^2)h_{12} - 2h_{12}^3$.

B.2 Isotropic spin-1 chains

The spin-1 representation of SU(2) is generated by

$$S^1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S^2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad S^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Unlike the SU(3) model to be discussed in the next section, none of the generators here has non-vanishing 1,3 entry. But this can be compensated by including powers of the generators in the Hamiltonian, such as

$$(S^+)^2 = \frac{1}{2}(S^1 + iS^2)^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore, it is customary to study the integrability of the class of models

$$H_1(\theta) = \sum_j \left(\cos\theta \vec{S}_j \cdot \vec{S}_{j+1} + \sin\theta (\vec{S}_j \cdot \vec{S}_{j+1})^2 \right), \tag{B.5}$$

as any SU(2) isotropic Hamiltonian of spin-s can be expressed in terms of a polynomial of degree $2s$. Solution of the Reshetikhin condition recovers the 6 integrable parameters, $\theta = \pm k\pi/4$, for $k = 1, 2, 3$, as recently confirmed by the exclusion of other possibilities [42].

In order to see the necessity to include the constant c in (1), we focus on the case $\theta = -\pi/4$, known as the Takhtajan-Babujian spin-1 model [6,7]. In terms of the Gell-Mann matrices

$$T^1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T^2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T^3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$T^4 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad T^5 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \tag{B.6}$$

$$T^6 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad T^7 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad T^8 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

its Hamiltonian after an overall rescaling can be written as

$$H_{\text{TB}} = \sqrt{2}H_1(\pi/4) = \sum_j \left(a_{\alpha\beta} T_j^\alpha T_{j+1}^\beta - \frac{4}{3} \right). \tag{B.7}$$

The non-vanishing coefficients are

$$a_{11} = a_{16} = a_{61} = a_{66} = a_{22} = a_{27} = a_{72} = a_{77} = -a_{44} = 1,$$

$$a_{33} = -\frac{1}{4}, \quad a_{38} = a_{83} = \frac{3\sqrt{3}}{4}, \quad a_{88} = \frac{5}{4}. \tag{B.8}$$

The above normalization of the Gell-Mann matrices have the structure constants

$$
f_3^{12} = \sqrt{2}, \quad f_7^{14} = f_6^{15} = f_6^{24} = f_7^{25} = f_5^{34} = -f_7^{36} = \frac{\sqrt{2}}{2}, \quad f_8^{45} = f_8^{67} = \frac{\sqrt{6}}{2},
$$
$$
d_8^{11} = d_8^{22} = d_8^{33} = -d_8^{88} = \frac{\sqrt{6}}{3}, \quad d_8^{44} = d_8^{55} = d_8^{66} = -d_8^{77} = \frac{\sqrt{6}}{6},
$$
$$
d_6^{14} = d_7^{15} = -d_7^{24} = d_6^{25} = d_4^{34} = d_5^{35} = -d_6^{36} = -d_7^{37} = \frac{\sqrt{2}}{2},
$$

(B.9)

with all the others not related by symmetry to the above being zero.

In this case, the coefficients in (21) become

$$
\begin{aligned}
u_{\alpha\beta\gamma} &= a_{\eta\zeta} a_{\iota\epsilon} a_{\delta\gamma} f_\theta^{\delta\epsilon} (f_\alpha^{\eta\iota} d_\beta^{\zeta\theta} + f_\beta^{\zeta\theta} d_\alpha^{\eta\iota}), \\
w_{\alpha\beta} &= a_{\eta\zeta} a_{\gamma\epsilon} a_{\delta\beta} f_\zeta^{\delta\epsilon} f_\alpha^{\eta\gamma}, \\
x_{\alpha\beta} &= a_{\gamma\zeta} a_{\gamma\epsilon} a_{\delta\beta} f_\eta^{\delta\epsilon} f_\alpha^{\zeta\eta}, \\
v_{\alpha\beta} &= y_\alpha = z_\alpha = 0.
\end{aligned}
$$

(B.10)

Obviously, the constant c will not matter for the Reshetikhin condition, as it disappears upon taking the commutator. However, it will enter the next order conditions (9). In fact, it can be checked using the `Mathematica` code provided in the supplementary information that at $m = 2$, (9) can only be satisfied if $c = 5\sqrt{2}/3$ in the original Hamiltonian $H_1(\pi/4)$ or equivalently $h_{j,j+1} = a_{\alpha\beta} T_j^\alpha T_{j+1}^\beta + 2$ in H_{TB} with the coefficients (B.8). This constant shift in the linear term of the Taylor expansion (1) corresponds to an overall factor that depends on the spectral parameter in the R-matrix for this model.

If the higher Reshetikhin conditions and the bootstrap program were used ignoring the constant shift, based on the belief that it should not affect the integrability of an interaction, then one could falsely arrived at the conclusion that an actual integrable Hamiltonian does not satisfy the YBE, despite fulfilling the Reshetikhin condition. The Takhtajan-Babujian model is the perfect example explaining this red herring.

B.3 SU(N) chains

The third example is the anisotropic SU(N) chain. The isotropic model with the full SU(N) invariance has been solved by Sutherland using the nested Bethe ansatz [43, 44]. In the basis

$$
(E^{\alpha,\beta})_{\gamma,\delta} = \delta_{\alpha\gamma}\delta_{\beta\delta},
$$

(B.11)

the Hamiltonian is given by

$$
H_{\text{SU}(N)} = \sum_j \sum_{\alpha,\beta=1}^N e_j^{\alpha,\beta} \otimes e_{j+1}^{\beta,\alpha}.
$$

(B.12)

Motivated by the partially integrable model studied in Ref. [8], the SU(N) symmetry can be broken to an $\mathfrak{S}_N$ permutation symmetry by introducing a diagonal potential

$$
H_{\mathfrak{S}_N} = \sum_j \sum_{\alpha=1}^N \left(\sum_{\beta\neq\alpha} e_j^{\alpha,\beta} \otimes e_{j+1}^{\beta,\alpha} + \Delta_\alpha e_j^{\alpha,\alpha} \otimes e_{j+1}^{\alpha,\alpha} \right).
$$

(B.13)

Reshetikhin's condition is only satisfied if $\Delta_\alpha = \pm 1$. This confirms the non-integrability of the Hamiltonian for generic Δ shown in Ref. [8] by the violation of the YBE.

References

[1] P. P. Kulish and E. K. Sklyanin, *Quantum spectral transform method recent developments*, In J. Hietarinta and C. Montonen, eds., *Integrable Quantum Field Theories*, pp. 61–119. Springer Berlin Heidelberg, Berlin, Heidelberg, ISBN 978-3-540-38976-7, doi:https://doi.org/10.1007/3-540-11190-5_8 (1982).

[2] M. P. Grabowski and P. Mathieu, *Integrability test for spin chains*, Journal of Physics A: Mathematical and General **28**(17), 4777 (1995), doi:10.1088/0305-4470/28/17/013.

[3] T. Kennedy, *Solutions of the yang-baxter equation for isotropic quantum spin chains*, Journal of Physics A: Mathematical and General **25**(10), 2809 (1992), doi:10.1088/0305-4470/25/10/010.

[4] K. H. Mutter and A. Schmitt, *Solvable spin-1 models in one dimension*, Journal of Physics A: Mathematical and General **28**(8), 2265 (1995), doi:10.1088/0305-4470/28/8/018.

[5] P. Bibikov, *How to solve yang–baxter equation using the taylor expansion of r-matrix*, Physics Letters A **314**(3), 209 (2003), doi:https://doi.org/10.1016/S0375-9601(03)00818-1.

[6] L. A. Takhtajan, *The picture of low-lying excitations in the isotropic heisenberg chain of arbitrary spins*, Physics Letters A **87**(9), 479 (1982), doi:https://doi.org/10.1016/0375-9601(82)90764-2.

[7] H. M. Babujian, *Exact solution of the isotropic heisenberg chain with arbitrary spins: Thermodynamics of the model*, Nuclear Physics B **215**(3), 317 (1983), doi:https://doi.org/10.1016/0550-3213(83)90668-5.

[8] Z. Zhang and G. Mussardo, *Hidden bethe states in a partially integrable model*, Phys. Rev. B **106**, 134420 (2022), doi:10.1103/PhysRevB.106.134420.

[9] L. Dolan and M. Grady, *Conserved charges from self-duality*, Phys. Rev. D **25**, 1587 (1982), doi:10.1103/PhysRevD.25.1587.

[10] L. Onsager, *Crystal statistics. i. a two-dimensional model with an order-disorder transition*, Phys. Rev. **65**, 117 (1944), doi:10.1103/PhysRev.65.117.

[11] M. de Leeuw, C. Paletta, A. Pribytok, A. L. Retore and P. Ryan, *Classifying nearest-neighbor interactions and deformations of ads*, Phys. Rev. Lett. **125**, 031604 (2020), doi:10.1103/PhysRevLett.125.031604.

[12] M. de Leeuw, C. Paletta, A. Pribytok, A. L. Retore and P. Ryan, *Yang-Baxter and the Boost: splitting the difference*, SciPost Phys. **11**, 069 (2021), doi:10.21468/SciPostPhys.11.3.069.

[13] L. Corcoran and M. de Leeuw, *All regular* 4 × 4 *solutions of the yang-baxter equation* (2024), 2306.10423.

[14] M. de Leeuw and V. Posch, *All 4 x 4 solutions of the quantum yang-baxter equation* (2025), 2411.18685.

[15] M. Yamaguchi, Y. Chiba and N. Shiraishi, *Complete classification of integrability and non-integrability for spin-1/2 chain with symmetric nearest-neighbor interaction* (2024), 2411.02162.

[16] M. Yamaguchi, Y. Chiba and N. Shiraishi, *Proof of the absence of local conserved quantities in general spin-1/2 chains with symmetric nearest-neighbor interaction* (2024), 2411.02163.

[17] J.-S. Caux and J. Mossel, *Remarks on the notion of quantum integrability*, Journal of Statistical Mechanics: Theory and Experiment **2011**(02), P02023 (2011), doi:10.1088/1742-5468/2011/02/P02023.

[18] M. G. Tetel'man, *Lorentz group for two-dimensional integrable lattice systems*, Soviet Journal of Experimental and Theoretical Physics **55**(2), 306 (1982), doi:http://jetp.ras.ru/cgi-bin/e/index/r/82/2/p528?a=list.

[19] B. Fuchssteiner, *Mastersymmetries, higher order time-dependent symmetries and conserved densities of nonlinear evolution equations*, Progress of Theoretical Physics **70**(6), 1508 (1983), doi:10.1143/PTP.70.1508.

[20] H. B. Thacker, *Corner transfer matrices and lorentz invariance on a lattice*, Physica D: Nonlinear Phenomena **18**(1), 348 (1986), doi:https://doi.org/10.1016/0167-2789(86)90196-X.

[21] J. P. Wang, *Representations of sl(2, ℂ) in category o and master symmetries*, Theoretical and Mathematical Physics **184**(2), 1078 (2015), doi:10.1007/s11232-015-0319-6.

[22] M. de Leeuw, A. Pribytok and P. Ryan, *Classifying integrable spin-1/2 chains with nearest neighbour interactions*, Journal of Physics A: Mathematical and Theoretical **52**(50), 505201 (2019), doi:10.1088/1751-8121/ab529f.

[23] A. Hokkyo, *Integrability from a single conservation law in quantum spin chains* (2025), 2508.20713.

[24] J. Lukierski, H. Ruegg, A. Nowicki and V. N. Tolstoy, *q-deformation of poincaréalgebra*, Physics Letters B **264**(3), 331 (1991), doi:https://doi.org/10.1016/0370-2693(91)90358-W.

[25] F. Bonechi, E. Celeghini, R. Giachetti, E. Sorace and M. Tarlini, *Inhomogeneous quantum groups as symmetries of phonons*, Phys. Rev. Lett. **68**, 3718 (1992), doi:10.1103/PhysRevLett.68.3718.

[26] C. Gomez, H. Ruegg and P. Zaugg, *Quantum deformed poincare algebra on a two-dimensional lattice*, Journal of Physics A: Mathematical and General **27**(23), 7805 (1994), doi:10.1088/0305-4470/27/23/023.

[27] P. Wang, *Discrete lorentz symmetry and discrete time translational symmetry*, New Journal of Physics **20**(2), 023042 (2018), doi:10.1088/1367-2630/aaaa17.

[28] P. Tempesta, *From symmetries to number theory*, Physics of Atomic Nuclei **72**(5), 866 (2009), doi:10.1134/S1063778809050172.

[29] C. Jordan, *Calculus of Finite Differences*, AMS Chelsea Publishing Series. Chelsea, ISBN 9781470441005 (1965).

[30] G. Boole and J. Moulton, *A Treatise on the Calculus of Finite Differences*, Dover books on advanced mathematics. Macmillan (1872).

[31] D. Levi, P. Tempesta and P. Winternitz, *Lorentz and galilei invariance on lattices*, Phys. Rev. D **69**, 105011 (2004), doi:10.1103/PhysRevD.69.105011.

[32] C. K. ZACHOS, *Umbral deformations on discrete space–time*, International Journal of Modern Physics A **23**(13), 2005 (2008), doi:10.1142/S0217751X08040548.

[33] D. Levi and P. Winternitz, *Continuous symmetries of difference equations*, Journal of Physics A: Mathematical and General **39**(2), R1 (2006), doi:10.1088/0305-4470/39/2/R01.

[34] P. Di Francesco, P. Mathieu and D. Senechal, *Conformal Field Theory*, Graduate Texts in Contemporary Physics. Springer-Verlag, New York, ISBN 978-0-387-94785-3, 978-1-4612-7475-9, doi:10.1007/978-1-4612-2256-9 (1997).

[35] J. Links, H.-Q. Zhou, R. H. McKenzie and M. D. Gould, *Ladder operator for the one-dimensional hubbard model*, Phys. Rev. Lett. **86**, 5096 (2001), doi:10.1103/PhysRevLett.86.5096.

[36] B. S. Shastry, *Exact integrability of the one-dimensional hubbard model*, Phys. Rev. Lett. **56**, 2453 (1986), doi:10.1103/PhysRevLett.56.2453.

[37] Y. Umeno, *Fermionic r-operator and algebraic structure of 1d hubbard model: Its application to quantum transfer matrix*, Journal of the Physical Society of Japan **70**(9), 2531 (2001), doi:10.1143/JPSJ.70.2531.

[38] M. Borsi, B. Pozsgay and L. Pristyák, *Current operators in bethe ansatz and generalized hydrodynamics: An exact quantum-classical correspondence*, Phys. Rev. X **10**, 011054 (2020), doi:10.1103/PhysRevX.10.011054.

[39] B. Pozsgay, *Algebraic construction of current operators in integrable spin chains*, Phys. Rev. Lett. **125**, 070602 (2020), doi:10.1103/PhysRevLett.125.070602.

[40] Z. Zhang, *Resonant valance bond and bethe ansatz on quasi-1d lattices* (2025), 2507.10471.

[41] F. Beukers, J. A. Sanders and J. P. Wang, *One symmetry does not imply integrability*, Journal of Differential Equations **146**(1), 251 (1998), doi:https://doi.org/10.1006/jdeq.1998.3426.

[42] H. K. Park and S. Lee, *Proof of nonintegrability of the spin-1 bilinear-biquadratic chain model*, Phys. Rev. B **111**, 134444 (2025), doi:10.1103/PhysRevB.111.134444.

[43] B. Sutherland, *Further results for the many-body problem in one dimension*, Phys. Rev. Lett. **20**, 98 (1968), doi:10.1103/PhysRevLett.20.98.

[44] B. Sutherland, *Model for a multicomponent quantum system*, Phys. Rev. B **12**, 3795 (1975), doi:10.1103/PhysRevB.12.3795.