Report for *SciPost* on Tachikawa and Zhang, "On a $\mathbb{Z}_3$-valued discrete topological term in 10d heterotic string theories"

This is a nice paper, which deserves publication. Let me try to restate the main result. There are two important heterotic string theories, called $E_8 \times E_8$ and $Spin(32)/\mathbb{Z}_2$, defined as having as their worldsheet theories the bosonic holomorphic VOAs of the same names. These VOAs have the same character; and so much of the physics of the string theories match. What the authors argue is that the effective actions do differ: specifically, in the difference of their effective actions is a $\mathbb{Z}_3$-valued discrete topological term. The language of the paper is heavily string theoretic, but then the authors supply a section for a reader who comes originally from mathematics, explaining the structure of their results in terms of (a conjectural description of) TMF. This referee found that section clarifying.

The authors had originally hoped to directly compute this difference as follows. There are maps $Spin(16) \times Spin(16) \to E_8 \times E_8$ and $Spin(16) \times Spin(16) \to Spin(32)/\mathbb{Z}_2$ (with slightly different kernels). Take the effective gauge theory actions for the two theories, and restrict along these maps. One finds two different effective gauge theory actions for $Spin(16) \times Spin(16)$. One could hope to directly compute the difference of these actions. The authors, sadly, report that this direct computation eluded them. Rather, the authors input more string theoretic knowledge and offer a less direct computation, which compactifies to 6D (along a single NS5-brane) and looks at an $SU(2)$ rotating the normal bundle, where the topological term can be seen visible. The abstract and introduction could be interpreted as suggesting that the authors supply two independent arguments for their result, one from $SU(2)$ and one from $Spin(16) \times Spin(16)$. I recommend clarifying that the results about the non-tachyonic non-supersymmetric $Spin(16) \times Spin(16)$ heterotic string are arrived at as corollaries of the authors calculation, and not as independent verification.

There is one place to be cautious about the authors' results: as a critical point (equations 2.20–2.22), there is a vital sign, and if a sign error was made, then the result would fail. I did not find any sign error, and I do not suggest that there is one. But signs are notoriously difficult to track, so this point makes me slightly nervous.

The following is a list of small typos and other comments.

1. In footnote 2, I have trouble parsing "the topological parts of the effective action ... depend continuously on the metric ...".
2. On page 3, "shrinks to zero side" should be "shrinks to zero size".
3. On page 4, "the H-filed" should be "the H-field".

4. On page 5, "is concerned the anomaly" should be "is concerned with the anomaly". I also suggest "we do need" in place of "we only need".

5. On page 6, when you write "Then our main result can be summarized as saying", it is not at all obvious yet why this is true. Of course, the point here is that you are outlining the rest of the paper, and I should wait for later explanation. But as written, I could have hoped that maybe you were doing the full translation to TMF now. Maybe a reference to the appropriate section would be appropriate?

6. On page 24, the sentence starting "The heterotic string theory we use" doesn't quite parse. The following sentence ends with a comma, but should end with a period.