Report on the paper "Local Zeta Functions of Multiparameter Calabi-Yau Threefolds from the Picard-Fuchs Equations" by Philip Candelas, Xenia de la Ossa, and Pyry Kuusela

In this paper, the authors attempt to generalize the the success on calculating local Zeta functions for Calabi-Yau threefolds, which was achieved in the recent paper by two of the present authors (Philip Candelas, Xenia de la Ossa) and Duco van Straten [1] (here we use the reference number of this paper). In the preceding work [1], a powerful method to calculate the local Zeta functions in terms of (power series expansion of) period integrals near special boundary points (MUM points) has been formulated. The work [1] was restricted to Calabi-Yau manifolds X with one-dimensional deformation $h^{2,1}(X) = 1$. In this paper, the authors present the basic formulas that appeared in [1] in more general settings so that they are applicable to multiparameter Calabi-Yau threefolds ($h^{2,1}(X) > 1$). Two examples of two-parameter Calabi-Yau manifolds and one example of (diagonal) five-parameter Calabi-Yau manifolds are studied at their special values of deformation parameters. In these examples, selecting a few special points, consistent results with the previous work are obtained. At the same time, the authors focus on the conifold locus and the locus of apparent singularities. Along the later locus, the authors find that the Frobenius matrix $U_p(\varphi)$ has a singularity, but this singularity can be moved by making a suitable change of variables. On the other hand, along the conifold locus, they report (in the first paragraph of page 36) that the singularity of $U_p(\varphi)$ behaves slightly different from [1] and they cannot determine the local Zeta functions. Also, the loci where Calabi-Yau manifolds get higher singularities are left open for a future problem.

The method for determining the local Zeta functions by using period integrals near MUM points is a great progress made in the reference [1]. This paper presents the construction of the Frobenius matrix $U_p(\varphi)$ for the Frobenius action $F_{p^n}(\varphi)$ in the general setting of variation Hodge structures and also in the form which is familiar to theoretical physicists. Thus the description provided in Sect. 2 and 3 for $U_p(\varphi)$ should be valuable for people to get into this research field.

As summarized above, I believe that this paper should be published in SciPost. Below are some comments/typos which should be taken into account before publication.

1. On page 27, the paragraph above Table 2. The equation (3.19) depends on n. Is it claimed that, for the values α_1, α_2 listed in Table 2, we have the relation (3.19) for all n? This is somewhat obscure in the present descriptions.

2. On page 28, below equation (4.5). "The roots of the quadratic factor ..." should be "The roots of the quartic factor ...".

3. Appendix B (Legendre family) is quite helpful for the reader. I have looked into some details, and below are comments based on them:

(3-1) In the theory of elliptic functions, the elliptic nome $q = e^{i\pi\tau}$ is defined by $\tau = i\frac{K(1-\lambda)}{K(\lambda)}$ and we have a series expansion of the lambda function $\lambda(\tau) = 16q - 128q^2 + \cdots = \frac{\theta_2(\tau)^4}{\theta_3(\tau)^4}$. As this relation shows, we should have

$$\begin{pmatrix} \int_A \Omega \\ \int_B \Omega \end{pmatrix} = \begin{pmatrix} \frac{2}{\pi}K(\lambda) \\ -2K(1-\lambda) \end{pmatrix}$$

for the period integrals. I do not think that the given form $\begin{pmatrix} \frac{2}{\pi}K(\lambda) \\ -2K(1-\lambda)+\frac{8\log 2}{\pi}K(\lambda) \end{pmatrix}$ is an integral basis. Of course, this difference of choice can simply be absorbed by the change $\alpha_p \to \alpha_p + (1-p)\log_p\left(\frac{1}{16}\right)$ in $U_p(\lambda) = E(\lambda^p)^{-1}\begin{pmatrix} 1 & 0 \\ \alpha_p & p \end{pmatrix} E(\lambda)$.

(3-2) Equation (B.4) should be $S_n(\lambda^p) = (\lambda^p - 1)^{n-2}$.

(3-3) The general form (or anzatz) $U_p(0) = u\begin{pmatrix} 1 & 0 \\ \alpha & p \end{pmatrix}$ is actually meant $U_p(0) = u_p\begin{pmatrix} 1 & 0 \\ \alpha_p & p \end{pmatrix}$, i.e., the choices of u and α depend on p (and also on n). As far as I checked (for the choice $n = 5$), to match the results from counting the number of rational points on E_λ, we need to set the parameters

	u_p	α_p
$p = 5$	1	1515
$p = 7$	-1	32039.
$p = 11$	-1	41888
$p = 13$	1	54769

Here we see that both parameters $u\,(u^2 = 1)$ and α depend on p. We can also observe that α_p depends on the choice of n.

For Calabi-Yau threefolds, as far as I understand from the text in (3.16), the corresponding parameters are found to be

$$u = 1, \quad \alpha_i = 0, \quad \gamma = \chi(\tilde{X}_t)\zeta_p(3)$$

for all p and n. This property seems to be in sharp contrast to the case of the Legendre family. Some remarks on this must be valuable for the reader to have good understanding about the results of this paper and also the preceding work [1].

(3-4) In case of the elliptic curve, as authors describe in the text (or in [1]), we can identify the sequence $\{a_p(E_\lambda)\}$ from $\det(1 - R_p(E_\lambda)T)$ with a cusp form of weight two. As far as I have checked for lower values of $\lambda = 2, 3, 4...$, we can identify them with $f_{32.2.a.a}, f_{96.2.a.a}, f_{48.2.a.a}, \cdots$ in the notation of LMFDB. It should be helpful for the reader to have these results (or some references to these results).

(4) When constructing the integral (local) solutions of Picard-Fuchs equations, the authors mention some relations to the so-called Γ-class citing the references

[28-32]. It should be noted that, for Calabi-Yau manifolds, the Γ-class is exactly the same as the central charge defined in Definition 2.1 and formulated as Conjecture 2.2 by S. Hosono in "Central Charges, Symplectic forms, and Hypergeometric Series in Local Mirror Symmetry", AMS/IP Stud. Adv. Math., 38, American Mathematical Society, Providence, RI, 2006, 405–439. I believe that it is appropriate to cite this paper, which was known before [29-32], and the integral structure of the solutions was discussed in general before them. The authors can find a concise summary of the paper in MathSciNet.