REPORT ON "LOCAL ZETA FUNCTIONS OF MULTIPARAMETER CALABI–YAU THREEFOLDS FROM THE PICARD–FUCHS EQUATIONS" BY PHILIP CANDELAS, XENIA DE LA OSSA, PYRY KUUSELA

The article at hand deals first and foremost with a generalization of the (still unpublished) preprint by Philip Candelas, Xenia de la Ossa and Duco van Straten (ref. [1] in the article). The latter provided a very efficient technique to compute local zeta functions for Calabi–Yau threefolds X with $h^{2,1}(X) = 1$. The computation starts directly from the solutions to the Picard–Fuchs system of a family of Calabi–Yau threefolds $\mathcal{X} \to B$, containing X as a special fiber. The main point is to view these solutions as series over the ring of p–adic numbers. The present article explains how to generalize this technique to $h^{2,1}(X) \geq 1$.

The authors review how the local zeta function of a fiber X_φ (under some mild assumption on $\mathrm{Pic}(X_\varphi)$) is determined by the characteristic polynomial of a certain matrix $U_p(\varphi)$. The main result is an explicit expression for this matrix.

It is explained that this matrix can be obtained from the solutions $\varpi(\varphi)$ of the Picard–Fuchs system of the family $\mathcal{X} \to B$ near a point of maximal unipotent monodromy, chosen to correspond to $\varphi = 0$, together with a conjectural form of the matrix $U_p(0)$ expressed in terms of the topological data of the mirror manifold $X^\vee$. Here, there is a new and intriguing relation to the Γ–class of $X^\vee$.

The idea to compute $U_p(\varphi)$ is to view the complex series $\varpi(\varphi)$ as p–adically convergent series, and then rely on the conjecture that the entries of $U_p(\varphi)$ are rational functions modulo p^n. Here is another new aspect: The authors give an explicit expression for the denominator of these rational functions in terms of the discriminant loci of the family $f : \mathcal{X} \to B$, the Wronskian matrix of $\varpi(\varphi)$ and the number n.

This technique is illustrated with three representative examples: The mirror octic in $\mathbb{P}(1, 1, 2, 2, 2)$, the mirror of a complete intersection two hyperplanes of degree $(1, 1)$ and $(1, 4)$, respectively, in $\mathbb{P}^1 \times \mathbb{P}^4$, both with $h^{2,1} = 2$, and the mirror Hulek–Verrill manifold with $h^{2,1} = 5$.

There are several important technical issues that arise and are explained in great detail with the help of these examples:

- Zeta functions for Calabi–Yau threefolds with conifold singularities
- Dependence on the choice of local coordinates of the base of the family $f : \mathcal{X} \to B$
- Dependence on the choice of local trivialization of the Hodge bundle $\mathrm{R}^3 f_* \mathbb{Z}$.

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The authors make an excellent effort in explaining (and simplifying) these mathematically challenging and possibly unfamiliar concepts, extracting from them a computational recipe, and providing a Mathematica package.

I recommend the manuscript for publication after the following points have been addressed:

(1) p.7, line 3 after eq. (2.5): missing "of"
(2) p.7, two lines before eq. (2.0): It should read "a point of maximal" instead of "the point of the maximal".
(3) p.9, beginning of Section 2.4: At this point of the article, an integral symplectic basis Π has not yet been introduced/defined, and a choice of such a basis has not yet been made. Therefore, at this point, it is not clear how the vectors Π and ϖ should be compared. So, at least, the definition of Π should be given. The comparison itself can be referred to reference [14].

This comparison does not uniquely fix the matrix ρ. One has in addition to impose $Y_{0ij} \in \{0, \frac{1}{2}\}$ as is stated in the explanation of the entries of the matrix ρ. Furthermore, to determine these constants one does not need require integral monodromies around *all* singular loci. It suffices to require integral monodromy around the maximal unipotent monodromy point under consideration.

As an aside, there is a general (conjectural) formula for the Y_{0ij}, see e.g. [1, §9.2]

$$Y_{0ij} \equiv -\frac{1}{2} Y_{iij} \bmod \mathbb{Z}$$

(4) p.15, line 1 in Section 3.2, again on p.17, two lines after (3.12): It should read "Frobenius".
(5) p.15, 4 lines before eq. (3.5): It should read "analogy".
(6) p.18, between (3.17) and (3.18): Twice a "relation above" is referred to. It would be better to refer to the equation (3.17). Also, there is a "we" missing in the first line after (3.17).
(7) p.25, the Picard–Fuchs equations in the middle of the page: The second equation should have a θ_2.
(8) p.33, line 7: The singularities of a two–parameter family are given by divisors, i.e. curves, in the moduli space. In the present case, these are the curves $\varphi_1 = 0$, $\varphi_2 = 0$, $\Delta = 0$, and if one chooses the natural compactification of the moduli space given by the secondary fan of the underlying toric variety $\mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^4}(-1,-1)^{\oplus 2})$, then there are also the curves $\psi_1 = \varphi_1^{-1} = 0$ and $\psi_2 = \varphi_2\varphi_1^{-1} = 0$ at infinity. Among all the points on these five curves parametrizing the singular fibers of the family, there are only two large complex structure points, namely $(\varphi_1, \varphi_2) = (0,0)$ and $(\psi_1, \psi_2) = (0,0)$. So, one cannot say that "any other singularities are large complex structure points". It is, however, correct that if $\mathcal{Y}(\varphi)$ is defined as in eq. (3.24) then $\mathcal{Y}(\varphi) = 1$.
(9) p.39, line 2. ... both η and $1/\eta$ *are p–adic integers* is a p–adic unit.
(10) p.39, line 3 after (A.3): It should read "Teichmüller".

Finally, I have tested the Mathematica package `CY3Zeta` and the instructions given in the Appendix E. The examples of the mirror quintic and the mirror octic work out perfectly. Then I have considered another simple example given by the following differential operator

$$\begin{aligned}\mathcal{L} = \theta^4 + \varphi \left(12500\,\theta^4 + 12500\,\theta^3 + 8125\,\theta^2 + 1875\,\theta - 120\right)\\ +1953125\,\varphi^2\theta\left(30\,\theta^3 + 60\,\theta^2 + 51\,\theta + 16\right)\\ +6103515625\,\varphi^3\theta\,(\theta+1)\left(20\,\theta^2 + 40\,\theta + 23\right)\\ +95367431640625\,\varphi^4\theta\,(\theta+2)\,(\theta+1)^2\end{aligned}$$

For this example, not all the steps to determine the matrix $U_p(\varphi)$ worked. I failed to find the rational matrix W and to find the coefficients α^i and $\hat{\gamma}$. It is very possible that I have made a mistake. Therefore, the corresponding Mathematica worksheet is attached to this report [1] I would be happy if the authors could point out the mistake to me and, if necessary, add a comment in the Appendix E to avoid this mistake.

As far as the rational matrix W is concerned, this can very easily be overcome by first computing this matrix directly from the solutions to the Picard–Fuchs equations instead of using the function `zFindW` and then using `zSetW`.

If there is no mistake, then the authors should explain in greater detail how to choose the values of `acc` and `maxdeg` in the function `zFindU0Constants` in general, and in particular for the operator $\mathcal{L}$. Alternatively, the authors should give additional conditions on the properties of the differential operators that are allowed to be studied with this package. These properties should then exclude the above operator $\mathcal{L}$. If $\mathcal{L}$ is not excluded, then it is probable that the function `zSingularityType` will have to be slightly modified. If the package is applicable to $\mathcal{L}$, then it could also be used to reproduce the polynomials $R_p(X_\varphi, T)$ in Table 12.1 of reference [44]. If this method works, then it should also yield the polynomials at K–points, an open question mentioned in Section 5.

Bibliography

[1] P. Mayr, "Phases of supersymmetric D-branes on Kahler manifolds and the McKay correspondence," JHEP **01** (2001), 018 doi:10.1088/1126-6708/2001/01/018 [arXiv:hep-th/0010223 [hep-th]].

[1]Uploading a file with the extension `.nb` was not allowed. Therefore I provide a pdf version and hope its contents can be easily copied into a Mathematica worksheet.

```
In[1]:= SetDirectory[$UserBaseDirectory];

In[2]:= << CY3Zeta.wl'

Out[2]= Null′

In[3]:= zSetNParameters[1]

In[4]:= zSetNMax[200]

In[5]:= zSetY[{Y[1, 1, 1] → 5}]

Out[5]= {Y[1, 1, 1] → 5}

In[6]:= zSetYhat[{Yhat[1, 1, 1] → 1 / 5}]

Out[6]= {Yhat[1, 1, 1] → 1/5}

In[7]:= zSetConifoldLocus[1]

Out[7]= 1

In[8]:= zSetOtherSingularLocus[1 + 3125 ϕ[1]]

Out[8]= 1 + 3125 ϕ[1]
```

The differential operator

```
In[9]:= Theta[f_, x_] := Theta[f, x] = x D[f, x]

In[10]:= L[f_] := 95 367 431 640 625 ((λ ϕ[1]) ^3 (1 / 3125 + λ ϕ[1]) ^4 D[f, {ϕ[1], 4}] / λ^4 +
      10 (λ ϕ[1] + 3 / 15 625) (λ ϕ[1]) ^2 (1 / 3125 + λ ϕ[1]) ^3 D[f, {ϕ[1], 3}] / λ^3 +
      24 λ ϕ[1] (1 / 3125 + λ ϕ[1]) ^2
       ((λ ϕ[1]) ^2 + 143 / 375 000 λ ϕ[1] + 7 / 234 375 000) D[f, {ϕ[1], 2}] / λ^2 +
      12 ((λ ϕ[1]) ^2 + 23 / 93 750 λ ϕ[1] + 1 / 117 187 500) (1 / 3125 + λ ϕ[1]) ^2
       D[f, ϕ[1]] / λ - 24 / 19 073 486 328 125 f) λ ϕ[1] / (1 + 3125 λ ϕ[1])

In[11]:= Factor[Coefficient[L[f[ϕ[1]]], D[f[ϕ[1]], {ϕ[1], 4}]]]

Out[11]=
ϕ[1]^4 (1 + 3125 λ ϕ[1])^3
```

Setting up the solution matrix

```
In[12]:= oo = 200

Out[12]=
200

In[13]:= ans = Sum[a[i] (λ ϕ[1]) ^i, {i, 0, oo}];

In[14]:= cfs = CoefficientList[Normal[Series[L[ans], {λ, 0, oo}]] /. λ → 1, {ϕ[1]}];

In[15]:= pi[0] = f[0] = ans /. Solve[Map[# == 0 &, cfs]][[1]] /. a[0] → 1;

In[16]:= cfs = CoefficientList[
    Normal[Series[L[f[0] Log[ϕ[1]] + ans], {λ, 0, oo}]] /. λ → 1, {ϕ[1]}];

In[17]:= f[1] = ans /. Solve[Map[# == 0 &, cfs]][[1]] /. a[0] → 0;
pi[1] = f[0] Log[ϕ[1]] + f[1];
```

In[19]:=
```
cfs = CoefficientList[
   Normal[Series[L[1 / 2 f[0] Log[ϕ[1]] ^ 2 + f[1] Log[ϕ[1]] + ans], {λ, 0, oo}]] /.
    λ → 1, {ϕ[1]}];
```

In[20]:=
```
f[2] = ans /. Solve[Map[# == 0 &, cfs]][[1]] /. a[0] → 0;
pi[2] = 5 (1 / 2 f[0] Log[ϕ[1]] ^ 2 + f[1] Log[ϕ[1]] + f[2]);
```

In[22]:=
```
cfs =
  CoefficientList[Normal[Series[L[1 / 6 f[0] Log[ϕ[1]] ^ 3 + 1 / 2 f[1] Log[ϕ[1]] ^ 2 +
        f[2] Log[ϕ[1]] + ans], {λ, 0, oo}]] /. λ → 1, {ϕ[1]}];
```

In[23]:=
```
f[3] = ans /. Solve[Map[# == 0 &, cfs]][[1]] /. a[0] → 0;
pi[3] = 5 (1 / 6 f[0] Log[ϕ[1]] ^ 3 + 1 / 2 f[1] Log[ϕ[1]] ^ 2 + f[2] Log[ϕ[1]] + f[3]);
```

In[25]:=
```
YhatRule = {Yhat[1, 1, 1] → 1 / 5};
```

In[26]:=
```
ωt[] = Table[f[i], {i, 0, 3}];
θωt[1] = Map[ExpandAll[Series[Theta[#, ϕ[1]], {λ, 0, oo}]] &, ωt[]];
θ2ωt[1] =
  Map[ExpandAll[Yhat[1, 1, 1] × Series[Nest[Theta[#, ϕ[1]] &, #, 2], {λ, 0, oo}]] &,
    ωt[]] /. YhatRule;
θ3ωt[] =
  Map[ExpandAll[Yhat[1, 1, 1] × Series[Nest[Theta[#, ϕ[1]] &, #, 3], {λ, 0, oo}]] &,
    ωt[]] /. YhatRule;
```

In[30]:=
```
θ2ωt[1][[4]] + O[λ] ^ 4
```

Out[30]=

$$-230\,\phi[1]\,\lambda + 777825\,\phi[1]^2\,\lambda^2 - \frac{67654319825}{12}\,\phi[1]^3\,\lambda^3 + O[\lambda]^4$$

The matrix E and its inverse

In[31]:=
```
zFindW[{4}, 100, 50]
```

No solution to the given accuracy

Out[31]=
```
{}
```

We find no solution. Therefore we construct the matrix W by hand

In[32]:=
```
Ylist = {1, 1, Yhat[1, 1, 1], Yhat[1, 1, 1]} /. YhatRule
```

Out[32]=

$$\left\{1, 1, \frac{1}{5}, \frac{1}{5}\right\}$$

In[33]:=
```
Emat = Table[
   ExpandAll[Series[Nest[Theta[#, ϕ[1]] &, pi[j], i] Ylist[[i + 1]], {λ, 0, oo}]],
   {i, 0, 3}, {j, 0, 3}];
```

In[34]:=
```
Wmat = ExpandAll[Emat.zσ.Transpose[Emat]];
```

In[35]:=
```
WMat = Factor[Normal[Wmat / Det[Wmat]] / Normal[1 / Det[Wmat]]]
```

Out[35]=

$$\left\{\left\{0,0,0,-\frac{i}{8\pi^3\,(1+3125\,\lambda\,\phi[1])^2}\right\},\right.$$
$$\left\{0,0,\frac{i}{8\pi^3\,(1+3125\,\lambda\,\phi[1])^2},-\frac{3125\,i\,\lambda\,\phi[1]}{4\pi^3\,(1+3125\,\lambda\,\phi[1])^3}\right\},$$
$$\left\{0,-\frac{i}{8\pi^3\,(1+3125\,\lambda\,\phi[1])^2},0,\frac{125\,i\,\lambda\,\phi[1]\,(3+15\,625\,\lambda\,\phi[1])}{8\pi^3\,(1+3125\,\lambda\,\phi[1])^4}\right\},$$
$$\left.\left\{\frac{i}{8\pi^3\,(1+3125\,\lambda\,\phi[1])^2},\frac{3125\,i\,\lambda\,\phi[1]}{4\pi^3\,(1+3125\,\lambda\,\phi[1])^3},-\frac{125\,i\,\lambda\,\phi[1]\,(3+15\,625\,\lambda\,\phi[1])}{8\pi^3\,(1+3125\,\lambda\,\phi[1])^4},0\right\}\right\}$$

We take this matrix as Wmat and try to compute the matrix E and its inverse using the given functions in CY3Zeta

In[36]:=
```
zSetW[Wmat]
```

In[37]:=
```
zComputeEMatrices[]
```

In[38]:=
```
zFindU0Constants[7, 6, 200]
zFindU0Constants[11, 6, 200]
```

Out[38]=

$$\{\alpha_1 \to \alpha_1[0] + 7\,\alpha_1[1] + 49\,\alpha_1[2] + 343\,\alpha_1[3] + 2401\,\alpha_1[4] + 16\,807\,\alpha_1[5],\ \gamma\text{hat} \to \gamma\text{hat}[0] + 7\,\gamma\text{hat}[1] + 49\,\gamma\text{hat}[2]\}$$

Out[39]=

$$\{\alpha_1 \to \alpha_1[0] + 11\,\alpha_1[1] + 121\,\alpha_1[2] + 1331\,\alpha_1[3] + 14\,641\,\alpha_1[4] + 161\,051\,\alpha_1[5],\ \gamma\text{hat} \to \gamma\text{hat}[0] + 11\,\gamma\text{hat}[1] + 121\,\gamma\text{hat}[2]\}$$

There seems to be no solution