REPORT: CÓRDOVA-COSTA-HSIN

1. Intro

This paper sets out to compute many important fusion rules for non-invertible symmetries that appear in discrete gauge theories. I think the paper assumes a basic understanding of the physical intuition that goes into defining such operators, and hence can be difficult for a more mathematical audience to read. I think it would be beneficial to make some things more precise for the mathematical audience. I would recommend this paper for publication provided that the following questions can be resolved.

- The authors state in the beginning of section 2 that discrete gauge theories can be defined by a path integral on the lattice, and the data needed to do so is determined by a cohomology class. By starting off with an invertible theory, classified by cohomology, one can gauge the global symmetry and obtain a gauge theory. What if the invertible theory is classified by a generalized cohomology theory, say for the fermionic setting? If the bordism invariant that detects the invertible phase is related to the η-invariant, how would a gauge theory be defined in that case using the η-invariant?
- In the language of Gaiotto and Johnson-Freyd, are surface operators the condensation monads? Does it describe the monad in terms of a condensation algebra in one category lower? I would like to know what is precisely meant by the"mesh" given in the introduction that describes a condensation defect? I would like the authors to try to make this statement mathematically precise, because condensations are used all over the paper but they have not been defined properly.
- Starting off in equation 2.4: could the authors comment on some categorical description for what it means to have a TQFT coefficient. Why does it necessarily have to be a TQFT i.e. a category that is non-degenerate, and not just a trivially graded category? Is it possible to give some references to the literature where fusion rules of 3-dimensional operators are studied in a more categorical point of view? That might be be helpful for understanding why the TQFT coefficient is necessary.
- In equation 2.13 what does it mean for operators of different dimension to act on each other. In particular how does D act on a Wilson line. It looks like the Wilson line is a module for D, but I am confused about the dimensions. How do you define what the category is that has both of these objects with this mutual action? I also do not see how D can be a morphism for W, since a morphism of lines would be a point operator.
- Can you provide more details on how you argued for the form of $\mathcal{Z}(G, \Sigma)$ in equation 3.10. Can you be more explicit about the operators in this theory? You use this relation for later computations e.g. equation 3.21. But then in what sense is the coefficient actually categorical, if you only consider the number it assigns in 3.21?
- In equation 2.16, why is the duality operator invertible intuitively? Shouldn't it fuse with respect to some (higher) Tambara Yamagami fusion relation?