REPORT ON “NONCOMMUTATIVE RESOLUTIONS AND CICY QUOTIENTS FROM A NON-ABELIAN GLSM” BY JOHANNA KNAPP, JOSEPH MCGOVERN

The article at hand discusses a particular Calabi–Yau threefold and its moduli space using the tools of the gauged linear sigma model (GLSM) and topological string theory at higher genus. This particular Calabi-Yau threefold is a free $\mathbb{Z}_3$ quotient of a complete intersection of three hypersurfaces of multidegree (1,1,1) in $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$. Its stringy Kähler moduli space has dimension one and has two large volume limits (one of which corresponds to the given Calabi–Yau threefold) and three further boundary points corresponding to (hyper)conifold points in the complex structure moduli space of the mirror manifold.

This geometry can be realized by a GLSM with the non–abelian gauge group $U(1)^3 \rtimes \mathbb{Z}_3$. The authors are mainly interested in the properties of the putative geometry at the second large volume limit, corresponding to the “negative phase” of the GLSM.

First, the properties of the “negative phase” of the GLSM turn out to be new and puzzling:

- It is non–regular, i.e. it has a Coulomb branch at infinity. This is the first example of a compact Calabi–Yau threefold with this property, before this had been only observed for K3 surfaces. The properties of such GLSM are not well–understood.
- It has a continuous non–abelian unbroken symmetry and a cubic superpotential, so in the IR there is a strongly coupled interacting gauge theory. The known methods for a quadratic superpotential do not apply here, so the authors leave it as a very interesting open problem to derive the putative geometry in this phase.

Second, and despite the issues just mentioned, using mirror symmetry and topological string theory (in the background of a topologically nontrivial flat B–field), they can nevertheless make a concrete proposal for the putative geometry in the IR. They propose that it is a noncommutative resolution of a complete intersection $\mathbb{P}(1,1,1,2,2,3)[4,6]$ with 63 nodal singularities, alternatively there is a non–Kähler small resolution with $\mathbb{Z}_3$ torsion in cohomology. The give evidence for this proposal by making several consistency checks which are completely convincing: Integrality of torsion–refined Gopakumar–Vafa invariants, and integrality of the monodromy around the second large volume limit in the integral symplectic basis.

As a side effect, this example adds to the (currently very small) list of noncommutative resolutions of nodal projective Calabi–Yau threefolds obtained from turning on fractional B–fields and the related torsion–refinement of Gopakumar–Vafa invariants, as first put forward by Katz, Klemm, Schimannek and Sharpe in 2022.

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The article is exceptionally clearly written and can serve as an introduction into this topic for beginners. I recommend it for publication after the following minor points (mostly typos) have been addressed:

(1) p.1, second paragraph, line 10: "too" should read "tool".
(2) p.2, line 2: The authors should give references for the "many more constructions" of non–birational Calabi–Yau threefolds and their descriptions in terms of GLSMs. There don't seem to be so many.
(3) p.2, paragraph around (1.1): The work of Batyrev and van Straten [1] should be emphasized here. They were the first to look at this example. Again around eqns. (3.6), (3.7).
(4) p.6, line 1 after (2.8): There is a typo in "we"
(5) p.12, eq. (2.36): The font of "t" should be the same as e.g. in (2.35)
(6) p.15, eq. (2.57): The exponent in the first equation should read $-\frac{\mathsf{t}}{3}$.
(7) p.15, Section 2.4: The "space" X appears here for the first time without having been defined. A definition is given later on top of of p. 24. This should be rearranged or reformulated.
(8) p.17, line 1 after (2.61): "is" should read "it".
(9) p.24, eqn. (3.21). The entries (1,2) and (2,1) of M should read $\frac{c_2}{24}$.
(10) p.25, last line of Section 3.2.1: Why just the first row of Table 2 ?
(11) p.25, Section 3.2.2, second paragraph: The (small) resolutions $\widehat{X}$ are not Calabi–Yau since they are not Kähler. But they do have vanishing first Chern class. This is why they are called small.
(12) p.26, line 2 after eq. (3.25): It should be pointed out that such an X_{def} exists due to a theorem of Namikawa and Steenbrink, see e.g. the first paragraph of Section 3.1 in reference [4].
(13) p.35, Conclusions: The authors should comment on whether and how the results of this article apply to the other examples in the work of Batyrev and van Straten [1]. They correspond to the differential operators AESZ 15 to AESZ 23. Of course, the case corresponding to AESZ 22 has been studied by Hosono, Takagi and Hori.
(14) p.41, line 8 after eq. (A.19): There is a typo in the second formula for the holomorphic limit.
(15) p.41, eq. (A.20): This gap condition was first observed in reference [36].

Bibliography

[1] V. V. Batyrev and D. van Straten, "Generalized hypergeometric functions and rational curves on Calabi-Yau complete intersections in toric varieties," Commun. Math. Phys. **168** (1995), 493-534 doi:10.1007/BF02101841 [arXiv:alg-geom/9307010 [math.AG]].