This is the referee report for the manuscript *An integrable deformed Landau-Lifshitz model with particle production?* by Marius de Leeuw, Andrea Fontanella, and Juan Miguel Nieto García, submitted to SciPost under the code `scipost_202507_00048v1`. We have not independently reproduced the computations, which we assume to be correct.

The authors formulate two classically integrable Landau–Lifshitz (LL) models, which they argue to respectively realise the continuum limits of the 'class-5 model' and the 'class-6 models' introduced in [1]. The LL model of the class-5 model is analysed in the main body of the manuscript, whereas the LL model of the class-6 model is treated more briefly in the appendix.

The class-5 and class-6 models are integrable quantum spin-1/2 chains defined through their respective R-matrices. Their Hamiltonians are one-parameter, non-Hermitian deformations of the Hamiltonian of the spin-1/2 XXX model. In the manuscript, following the prescription of [2], the authors derive the action of the LL model as a continuum limit of the action in the representation of the partition function quantum spin-1/2 chain as a path integral over coherent states. The limit is supplemented by a rescaling of the deformation parameter. To prove that the LL models corresponding to the class-5 and class-6 models are classically integrable, the authors use the results of [3] to argue for the existence of an infinite set of independent conserved charges in involution with the Hamiltonian and among themselves. The authors perturbatively quantise the LL model corresponding to the class-5 model around a constant solution to the equations of motion. They compute the S-matrix one-to-two-particles process at tree level, thereby highlighting the presence of particle production in the model.

In our opinion, the manuscript often fails to provide sufficiently convincing arguments to justify the significance of the model and of the results, whose scope is at times overstated. We list below the points on which we are critical.

1. Since the derivation of the LL model as a continuum limit of the class-5 model relies on a heuristic path integral, a consistent matching between the integrable structures of the LL model and the class-5 model is imperative to substantiate the correspondence between the two models.

   (a) The authors present the Lax matrix of the spin-s class-5 model in (1.12). (They do not compute the Lax matrix of the class-6 model.) They promote (1.12) to one component of the Lax connection of the LL model in (2.24); however, they do not succeed in computing the second component of the Lax connection. This absence casts doubts on the status of the LL model as a genuine continuum limit of the class-5 model as it prevents an additional comparison with the (defective) transfer matrix of the class-5 model computed in [4].

   (b) The authors do not explore the possibility of promoting the first Lax matrix in (1.7), which satisfies the RLL-relation only for the fundamental spin-1/2 class-5 model, to a component of a Lax connection that would allow for the construction of a classical monodromy matrix.

   (c) The authors rely on [3] to justify the existence of an infinite set of independent conserved charges in involution. Nonetheless, they do not attempt to match the conserved charges of the LL models with those of the class-5 and class-6 models. They could have applied the boost operator to the infinite quantum spin-1/2 chains in order to generate conserved charges of the class-5 and class-6 models, and sought to identify their counterparts in the classically integrable LL model.

2. The authors perturbatively quantise the LL model of the class-5 model around a constant solution to the equations of motion to show the presence of particle production. The reasoning behind resorting to a perturbative quantisation is not clear if the integrable quantum spin-1/2 chain is already available.

   (a) The authors justify the appearance of particle production by arguing that states with different particle numbers but identical energy arise within the Jordan cells of the Hamiltonian of the class-5 model. Moreover, they claim that the term in the Lagrangian density responsible for particle production is removable by a change of basis in the Hilbert space of the underlying quantum model via the Drinfeld twist of the class-5 model. However, the counterparts of both the Jordan cells of the Hamiltonian and the Drinfeld twist within the perturbative quantisation of the LL model are far from clear. This point further supports the need for a direct analysis of the scattering problem in the class-5 model realised as an infinite quantum spin-1/2 chain.

   (b) For the analysis of the scattering problem of the LL model to include a complete account of the standard hallmarks of integrability, the factorisability of the S-matrix of the three-to-three-particles process must be examined (rather than merely suggested as a future line of research under the point 'S-matrix factorisation' on page 21 of the conclusion) and a characterisation of the allowed scattering processes must be provided.

3. Since the class-5 and class-6 models, together with their purported LL models, lack any demonstrable connection to Yang–Baxter deformations and do not exhibit novel traits, the assertion of their potential relevance to the AdS/CFT correspondence is arguably unfounded.

   (a) Both non-diagonalisable Hamiltonians of quantum spin-1/2 chains and particle production have already been observed in well-established integrable models; hence, the role of the class-5 model as a testing ground for such mechanisms is called into question. In addition, within the AdS/CFT correspondence, as the authors note, integrable quantum spin-1/2 chains already appear, for instance, in the fishnet theory [5], whilst particle production has been observed in the light-cone quantisation of a Yang–Baxter deformation of the non-linear σ-model on $\mathrm{AdS}_5 \times \mathrm{S}^5$ in [6].

   (b) On pages 2–3 the authors write:

   > In addition to their importance by itself, the Class 5 model has a connection to the topic of Yang-Baxter deformations in AdS/CFT. The LL action played an important role as a non-trivial check of the duality between strings in $\mathrm{AdS}_5 \times \mathrm{S}^5$ and $\mathcal{N} = 4$ SYM. [...] As the Class 5 model can be written as a Jordanian Drinfeld twist of the XXX spin chain, it provides a simpler toy model where many of the unconventional characteristic of Jordanian deformations and non-diagonalisable spin chains can be studied (for example, particle production despite having integrability [46]).

On top of the previous objections, it should also be clarified that the customary role played by the LL model in the AdS/CFT correspondence is alien to the class-5 and class-6 models. Specifically, the LL model provides an effective action in the semi-classical limit of large quantum numbers, which enables a quantitative comparison between the energies of classical solutions and the conformal dimensions of primary operators within entire closed sectors, thereby bypassing the need to analyse specific individual cases.

(c) On page 3 of the introduction the authors write:

> Another situation where integrability and non-diagonalisability meet is in the context of non-relativistic strings. [...] However, the classical monodromy matrix associated to this Lax connection does not fit the standard form: it is non-diagonalisable and the eigenvalues are independent of the spectral parameter [50]. Although the class-5 model is a quantum theory, it also has a non-diagonalisable transfer matrix, so it might shed some light on this problem.

Without a classical counterpart of the defective transfer matrix for the class-5 model, such as a potential transfer matrix for the LL model, the claim that the class-5 model could shed light on the non-relativistic classical transfer matrix lacks conceptual foundation.

(d) For the aforementioned reasons, we believe that the future lines of research suggested under the points 'Drinfeld twist' and 'Non-relativistic limits' on page 20 of the conclusion are too vague and require further concretion.

Although the manuscript is generally well written, certain passages are cluttered in a way that hinders the clarity of the exposition.

4. After writing the R-matrix of the class-5 model in (1.1) on page 4, the authors comment on certain simplifications that 'will be used in the paper'. Despite their limited importance to the remainder of the work, these simplifications are never clearly implemented in the subsequent analysis.

5. The first Ansatz for the Lax matrix in (1.7) on page 5 is based on its identification with the R-matrix in (1.6) on page 5. Nonetheless, the latter is inexplicably shifted with respect to the former.

6. The introduction of the spin variables and of the generalised derivatives in Section 2 seems to have no impact on the subsequent analysis of the LL model.

7. In Subsection 2.3, it is conceivable that the authors intended to identify the boost operator with a recursion operator and with a Nijenhuis operator, but if such an identification is indeed correct, it is not made sufficiently explicit. In any case, the presentation of strong symmetries as given appears to be of only marginal relevance to the integrable structure of the LL model.

Finally, we would like to raise some minor points.

9. On page 2 of the introduction, the authors write:

> In this article, we study the continuum limit of the "Class 5" model, described in [1] in a search for all possible two-dimensional integrable spin chains.

We understand that 'two-dimensional' refers to the dimension of the local space of the integrable models. To avoid ambiguity, it would be preferable to replace 'two-dimensional integrable spin chains' with the standard 'integrable spin-1/2 chains'.

10. Reference [46] is cited twice in the same sentence on page 3 of the introduction, in reference to the same point, which is most likely a typographical error.

11. It would be desirable for the authors to provide a reference in which the definition of the Drinfeld twist in (1.10) on page 6 is shown to be equivalent to the definitions employed in other references on integrable models; see, for example, Subsection 3.3 of [7].

12. The reader would benefit if the notation used in Subsection 2.3 were explained in greater detail.

For the reasons detailed above, we do not believe that the manuscript, in its current state, opens a new pathway in an existing or a new research direction with clear potential for substantial follow-up work. Unless the authors satisfactorily address the critiques raised, we cannot recommend the publication of the manuscript.

References

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