Report on "Complexity of PXP scars revisited"

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October 2025

In this paper, the Authors study Krylov complexity of states in the Schrödinger picture, often also referred to as spread complexity, in the context of the PXP model. This is a quantum many-body system that is known to feature mostly non-integrable dynamics, but whose Hilbert space contains some states, dubbed quantum scars, with non-thermalizing dynamics. In particular, the self-fidelity of these states displays some quasi-periodic features known as "revivals". The goal of the Authors is to study the Krylov complexity of known scar states in the PXP model (in particular, states belonging to the class of the Néel-like states denoted as $|Z_k\rangle$) in order to understand this phenomenology from the perspective of the Krylov space structure and the complexity dynamics. The article is presented as a response to references [44,45], which study the same problem via analyses with which the Authors do not fully agree.

The Authors find that the Lanczos coefficients of the $|Z_k\rangle$ states under consideration start following an almost-integrable profile dominated by symmetry (the arch), and then transition to a slow descent regime (which they refer to as the buttress) that eventually covers the full Krylov space. They explain these regimes by conveniently rewriting the PXP Hamiltonian H_{PXP} as the sum of two terms: A free term $H_{\mathrm{PXP,lin}}$ controlled by generators of the $sl(3,\mathbb{C})$ algebra, and an integrability-breaking term $H_{\mathrm{PXP,res}}$. The Authors present numerical and analytical arguments that associate the first regime of the Lanczos coefficients to $H_{\mathrm{PXP,lin}}$ and they interpret the corresponding Krylov subspace as the non-thermalizing sector of the system's scarred Hilbert space; they additionally argue that the eventual breakdown of the arch regime is a consequence of the overlap of the initial state with the non-thermalizing Hilbert subspace, to which it eventually leaks during its time evolution. The Authors present their analysis of the arch as an improvement over [44,45], where the symmetry-controlled regime of the Lanczos coefficients was, respectively, fitted to a q-deformation of the $su(2)$ algebra and described via a forward-scattering approximation, but not derived from first principles as an outcome of the structure of the Hamiltonian, as presented in this paper.

I think the formal symmetry analysis in this paper is very convincing. Besides this, there are some minor inaccuracies in the paper that would benefit from a slight revision and, more importantly, I have some concerns on the reliability of their numerical results. I will elaborate on all these points below.

1. Spread complexity $\mathcal{C}_K(s)$ is presented in equation (3) as the minimum complexity at time s obtained by optimizing over all the complexities $\mathcal{C}_{\mathcal{B}}(s)$ defined via other choices of orthonormal basis $\mathcal{B}$. This seems slightly inaccurate. To my understanding, the actual statement proved in reference [6] is the following: For every orthonormal basis $\mathcal{B}$ there exists a $\mathcal{B}$-dependent time $s_{\mathcal{B}}$ such that $\mathcal{C}_K(s) < \mathcal{C}_{\mathcal{B}}(s)$ for $s \in [0, s_{\mathcal{B}})$. Crucially, the range of s for which $\mathcal{C}_K(s) < \mathcal{C}_{\mathcal{B}}(s)$ holds is $\mathcal{B}$-dependent. However, in equation (3) the Authors are holding s fixed independently from $\mathcal{B}$: This does not work, not even if s belongs to some finite interval as the Authors claim below equation 3, because there always exists a choice of $\mathcal{B}$ that can push $s_{\mathcal{B}}$ arbitrarily close to zero. One can understand this by considering a counterexample already provided in [6]: For every time $s = s'$ the basis $\mathcal{B}$ constructed such that its first basis element is $|\Psi(s')\rangle$ will have the minimal complexity value (equal to zero) at that particular instant of time $s = s'$.

2. In order to assess the structure of the Krylov space of a state $|Z_k\rangle$ under the action of the total Hamiltonian $H_{\text{PXP}} = H_{\text{PXP,lin}} + H_{\text{PXP,res}}$, the Authors consider the Krylov subspaces $\mathcal{H}_{\text{PXP,lin}} = \left\{H^n_{\text{PXP,lin}}|Z_k\rangle\right\}_{n\geq 0}$ and $\mathcal{H}_{\text{PXP,res}} = \left\{H^n_{\text{PXP,res}}|Z_k\rangle\right\}_{n\geq 0}$ (where the initial state dependence is loosely dropped), associating the former to the non-thermalizing Hilbert subspace, and the latter to the thermalizing one. It should be noted, however, that the full Krylov space $\mathcal{H}_{\text{PXP}} = \left\{\left(H_{\text{PXP,lin}} + H_{\text{PXP,res}}\right)^n |Z_K\rangle\right\}_{n\geq 0}$ contains cross terms terms of the form $H^m_{\text{PXP,lin}}\, H^n_{\text{PXP,res}}|Z_k\rangle$ with $m, n > 0$ which do not seem to belong to either $\mathcal{H}_{\text{PXP,lin}}$ or $\mathcal{H}_{\text{PXP,res}}$ because $[H_{\text{PXP,lin}}, H_{\text{PXP,res}}] \neq 0$. Therefore the proposed decomposition $\mathcal{H}_{\text{PXP}} = \mathcal{H}_{\text{PXP,lin}} \bigoplus \mathcal{H}_{\text{PXP,res}}$ does not seem to be correct. For the identification of $\mathcal{H}_{\text{PXP,lin}}$ with $\mathcal{H}_{\text{non}-\text{thermal}}$ and $\mathcal{H}_{\text{PXP,res}}$ with $\mathcal{H}_{\text{thermal}}$ to be consistent with a Hilbert space decomposition of the form $\mathcal{H}_{\text{thermal}} \bigoplus \mathcal{H}_{\text{non}-\text{thermal}}$, one needs the thermal and non-thermal Hilbert spaces to be disconnected, as the Authors review in section 3.1, but the extra sector in the Krylov space $\mathcal{H}_{\text{PXP}}$ generated by the aforementioned cross terms seems to potentially spoil this picture. Could the Authors comment on this?

3. The Authors compute an upper bound on the Krylov space dimension as the number of product states without adjacent 1's, which is given by the Lucas numbers. In their numerics, they compare the resulting Krylov space dimension $\dim \mathcal{H}_{\text{PXP}}$ to this upper bound and find that it is almost saturated at large system sizes, but they do not have an a priori estimate of what the actual value of $\dim \mathcal{H}_{\text{PXP}}$ should be. I am therefore concerned that the almost maximal Krylov dimension that they obtain might be affected by a not properly cured numerical instability in their application of the Lanczos algorithm. In particular, in cases where the Lanczos algorithm is applied to an initial state with a sub-maximal Krylov space dimension, the authors of arXiv:2505.02670 have recently shown that there exists an

extra source of numerical error, not cured by the re-orthogonalization algorithms, which consists on the accidental generation of spurious extra orthogonal directions in the successive iterations or the algorithm. As a result of this, one might obtain a fake maximal Krylov space dimension. I am worried that this instability might be affecting the buttress regime of the Lanczos coefficients presented in this paper, and I think that the numerical analysis would be improved if the Authors showed that this source of instability has been avoided in their numerics (by e.g. performing consistency checks on the obtained Lanczos coefficients, or by giving an estimate -rather than a bound- of the expected Krylov dimension, if this is doable), or if they corrected it (by e.g. increasing machine precision, or implementing the suggestions given in arXiv:2505.02670) in case they do find that the instability has affected their results.

4. It would be enlightening for the reader if the Authors discussed the behavior of complexity in figures 14 and 15 in relation to the Krylov space dimension (which bounds $\mathcal{C}_K(t)$ from above) in each of the cases plotted.

I think this paper is an interesting symmetry-based explanation of the non-thermalizing properties of scar states in the PXP model from the perspective of Krylov methods and complexity, and I would be happy to recommend it for publication once the Authors have addressed the concerns that I have raised, which are all minor issues except for the potentially more relevant closer inspection of their numerical results.