Summary. The manuscript studies quantum quenches in the PXP model, focusing on the time evolution of spread complexity and related Krylov-basis diagnostics for both scarred and thermalizing product initial states. The authors report a characteristic "arch" structure in Lanczos coefficients b_n followed by a "buttress" regime, relate them to scarred dynamics, and propose an explanation based on decomposing the PXP Hamiltonian into a "linear" part $H_{\mathrm{PXP,lin}}$ and a "residual" part $H_{\mathrm{PXP,res}}$ using representation theory of $\mathfrak{sl}_3(\mathbb{C})$.

I find the algebraic perspective potentially interesting. However, several central claims are currently not sufficiently supported by numerical evidence, and the relation to the extensive existing literature on PXP scarring seems incomplete. Moreover, the manuscript is very long. While having all derivations is helpful, the central message is very hard to extract.

In view of two major deficiencies above we do not recommend the publication of the manuscript in its current form. The following points should be addressed:

1. **Structure of the paper.** We suggest to either move some of the longer derivations to appendices or to make a separate concise section with the summary of results that presents the most important equations and discusses the new conceptual results of this work. In addition, the subsection names that use single words ("The arch", "Inferences", etc) can be expanded to become more descriptive.

2. **Central hypothesis needs quantitative validation.** A key hypothesis is that the Krylov subspace generated by $H_{\mathrm{PXP,lin}}$ captures the scarred dynamics, while $H_{\mathrm{PXP,res}}$ is responsible for thermalization. The paper argues that $H_{\mathrm{PXP,lin}}$ admits an SGA-like construction and would therefore yield revivals with period $T = \sqrt{2}\,\pi$, but it is not demonstrated numerically that $H_{\mathrm{PXP,lin}}$ indeed *dominates* the revival dynamics of $|Z_2\rangle$ (and $|Z_3\rangle$), nor that $H_{\mathrm{PXP,res}}$ does not significantly spoil these revivals already at early times.

   In my view, sufficient numerical evidence is missing to support this picture. To address this issue one could compare dynamics under $H_{\mathrm{PXP,lin}}$ alone with dynamics of the full H_{PXP}. In the same spirit, the role of the Krylov subspace $\mathcal{K}_{\mathrm{PXP,lin}}$ defined in Eq. (46) should be quantified: does time evolution projected to $\mathcal{K}_{\mathrm{PXP,lin}}$ approximately reproduce the full dynamics until times where revivals are still good?

3. **Relation to established constructions (FSA, SU(2), parent Hamiltonians) is unclear.** How does $\mathcal{K}_{\mathrm{PXP,lin}}$ (Eq. (46)) relate to the forward-scattering-approximation (FSA) subspace introduced in Ref. [31] (Nature Physics **14**, 745 (2018))? Also the relation to the SU(2) / algebraic viewpoints in Ref. [70] (Phys. Rev. Lett. **122**, 220603), and to the spin-1 mappings / parent-Hamiltonian approaches (e.g. 10.1103/PhysRevA.107.023318, 10.1103/PhysRevX.13.011033) should also be discussed. A short discussion of what is new/complementary relative to these works would improve the positioning of the paper.

4. **Section 5.3 ("Evidences") is not compelling as evidence for the main hypothesis.** Fig. 8 mainly reiterates known fidelity revivals for $|Z_2\rangle$ versus $|Z_4\rangle$, and Fig. 9 reports overlaps with eigenstates that have as well been extensively reported in previous literature (in a more compelling way); the discussion around Fig. 10 is likewise not sufficient to substantiate the paper's central mechanism, since truncating the Hamiltonian and inspecting level spacings in that truncated model does not by itself establish that $H_{\mathrm{PXP,lin}}$ controls the revival dynamics or that $H_{\mathrm{PXP,res}}$ is responsible for thermalization. It could be insightful to inspect the proposed decomposition by, e.g., comparing spectral properties of $H_{\mathrm{PXP,lin}}$ to those of $H_{\mathrm{PXP,res}}$ and especially H_{PXP}, and by tying these comparisons to dynamical diagnostics that directly separate their roles (see point 1); finally,

the use of $L = 12$ throughout this section is quite small and finite-size effects could be significant. For any results to be meaningful they have to be validated across different system sizes.

5. **Broader set of scarred product states should be discussed (and ideally tested).** Recent work (10.1103/PhysRevLett.134.160401) reports additional low-entanglement scarred states with 3-site and 4-site unit cells. These should at least be mentioned for completeness in the discussion of scarred dynamics in PXP, including how they relate to the product states $|Z_3\rangle$ and $|Z_4\rangle$ (for this see that work's supplemental material). More importantly, these states provide a stress test for the present framework. A numerical analysis analogous to the ones proposed for $|Z_2\rangle$ would help validate reported connection between "arch/buttress" and scarring.

6. **Definitions of "arch" and "buttress" should be sharpened.** I don't see where the notions of "arch" and "buttress" are rigorously defined, so it is unclear how they are identified from the numerical data (especially in cases where it is not visually apparent). Please give a clear criterion for where the arch ends and the buttress begin (or make it more prominent in the paper if it has been done already).

7. **Presentation of figures can be improved.** Apart from the fact that most numerical simulations in this paper just show scarred dynamics of the PXP model in different forms (most of which have been already reported extensively in the literature) and don't provide strong evidence for the reported claims, as discussed before, also the presentation of the numerical plots could be improved. Some examples (non exhaustive list) are the following: (i) In Fig. 8, using a shared y-axis (and consistent limits) across panels would facilitate direct comparison of revival amplitudes. (ii) In Fig. 9, a 2D scatter plot with energy on one axis and overlap magnitude on the other would better convey the separation between special eigenstates and the bulk; the current 1D color representation does not clearly show the bulk structure and will become hard to read for larger L. (iii) Fig. 10: the values on the energy-axes are extremely small (consistency in label size would be generally good) and the captions (especially for panel (d)) should be clarified; the color choices in (d) seem to create an unintended connection to panels (a)-(c) (in which the same two colors are used for seemingly different things).

8. **Definition of spread complexity and the minimization in Eq. (3) needs clarification.** Taken literally, the minimization over all bases in Eq. (3) would allow choosing a basis that contains $|\Psi(s)\rangle$ as one element, making $C_B(s)$ trivially minimal. This has been already discussed in Ref. [6] (10.1103/PhysRevD.106.046007, 2202.06957) in terms of a functional minimization. The manuscript should adjust the definition so that Eq. (3) is well-defined and consistent with Ref. [6].

9. **Comment on Eq. (16) (dimension of non-thermal subspace) appears misleading.** The statement "When the dimension of the non-thermal subspace $\mathcal{H}_{\text{non-thermal}}$ is not exponentially small, relatively to the dimension of the thermal subspace $\mathcal{H}_{\text{thermal}}$, it is possible to observe the non-thermal phenomena in experiments." seems too strong. Scarred PXP dynamics have been experimentally observed (Nature **551**, 579–584 (2017)) even though the scarred sector is exponentially small compared to the thermal part of the Hilbert space.

10. **Minor comment: wording around Eq. (17).** The sentence "... within the subspace $|\mathcal{E}\rangle$, ..." is imprecise: $|\mathcal{E}\rangle$ denotes a single eigenstate, not a subspace.